Lecture 9 – The First and Second Derivative Rules

MATH 1400: Calculus for Social and Life Sciences I
Section 2.2

Contents

1 Review: Describing a Graph 1

2 First Derivative Rule 3

3 Second Derivative Rule 5

4 Connections Between a Function and Its Derivative 7

5 Recognizing Behavior and Concavity 10

1 Review: Describing a Graph

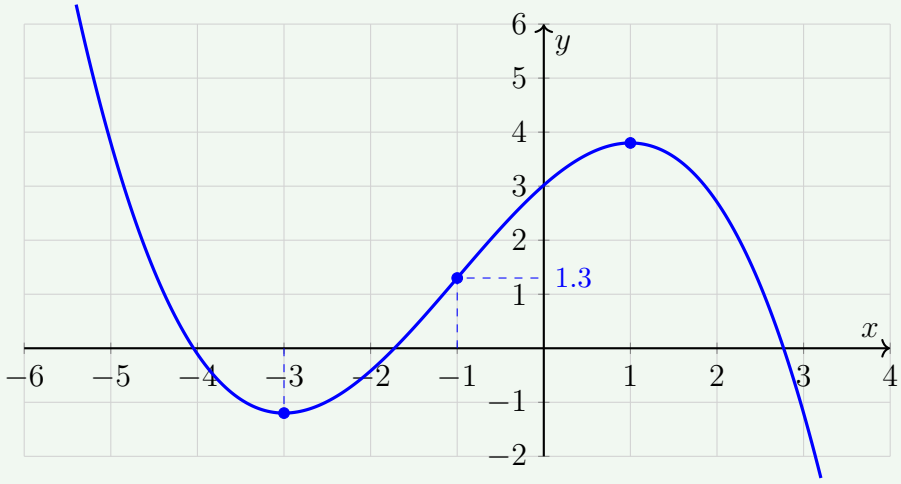
Recall the main features used to describe a graph.

Quick Summary

- **Increasing/decreasing:** The graph goes up/down from left to right.
- **Relative maximum/minimum:** The graph changes from increasing to decreasing, or decreasing to increasing.
- **Absolute maximum/minimum:** The largest/smallest function value on its entire domain.
- **Concave up/down:** The slopes increase/decrease as we move left to right.
- **Inflection point:** A point where concavity changes.
- **Intercepts:** At an x-intercept, $y = 0$; at a y-intercept, $x = 0$.
- **Asymptotes:** Lines approached by a graph (or along which it becomes unbounded, for vertical asymptotes).

Example 1: Describing a Graph

Consider the graph below and answer the following questions.



(a) On which interval(s) is the function increasing?

(b) On which interval(s) is the function decreasing?

(c) Where is/are the relative maximum point(s)?

(d) Where is/are the relative minimum point(s)?

(e) What is the absolute maximum value?

(f) What is the absolute minimum value?

(g) On which interval(s) is the function concave up or down?

(h) What is/are the inflection point(s)?

(i) What is/are the x-intercept(s)?

(j) What is/are the y-intercept(s)?

(k) What is/are the undefined point(s)?

(l) What is/are the asymptote(s)?

Solution (a)–(b): Increasing and Decreasing

The graph rises between $x = -3$ and $x = 1$ and falls outside that interval.

$$\boxed{\text{Increasing: } (-3, 1)}$$

$$\boxed{\text{Decreasing: } (-\infty, -3) \text{ and } (1, \infty)}$$

Solution (c)–(d): Relative Extrema

At $x = 1$ the graph changes from increasing to decreasing; at $x = -3$ it changes from decreasing to increasing.

$$\boxed{\text{Relative maximum at } x = 1, \quad \text{relative minimum at } x = -3.}$$

Solution (e)–(f): Absolute Extrema

The curve continues upward to the left and downward to the right. It has **no absolute maximum or absolute minimum** on $\mathbb{R}$.

Solution (g)–(h): Concavity and Inflection

The slopes increase to approximately $x = -1$, then decrease.

$$\boxed{\text{Concave up: } (-\infty, -1), \qquad \text{Concave down: } (-1, \infty).}$$

The approximate inflection point marked on the graph is $\boxed{(-1, 1.3)}$.

Solution (i)–(l): Intercepts, Undefined Points and Asymptotes

The displayed curve has approximately three x-intercepts:

$$(-4.044, 0), \quad (-1.725, 0), \quad (2.769, 0).$$

Its y-intercept is $\left(0, \frac{483}{160}\right) \approx (0, 3.019)$. The polynomial has no undefined points or asymptotes.

2 First Derivative Rule

The derivative describes the **slope of the tangent line**.

First Derivative Rule

On an interval where f is differentiable:

- If $f'(x) > 0$ throughout the interval, f is **increasing** there.
- If $f'(x) < 0$ throughout the interval, f is **decreasing** there.

If $f'(a) = 0$, the tangent line at $x = a$ is **horizontal**; this alone does not establish a maximum or minimum.

Example 1: Sketching from the First Derivative

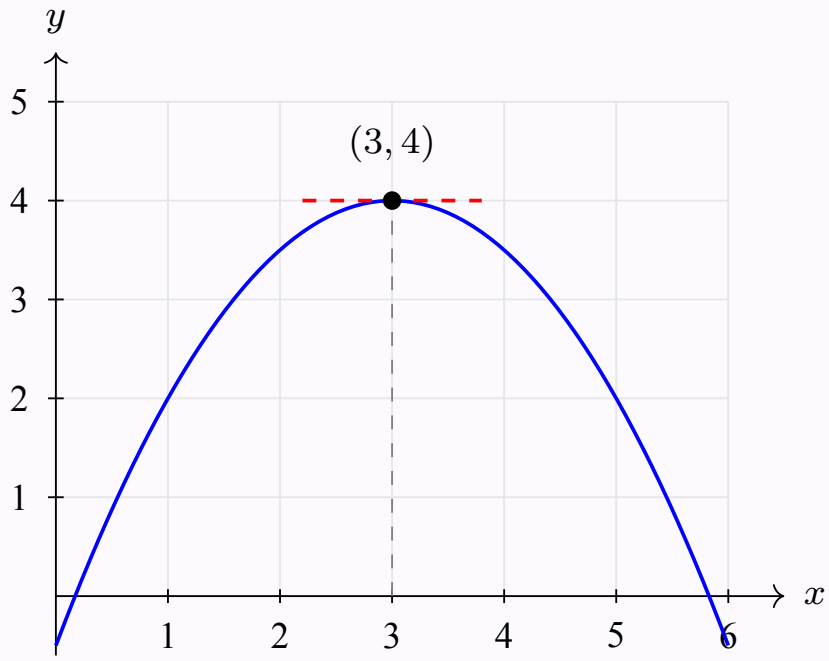
Sketch the graph of a function $f(x)$ satisfying:

(a) $f(3) = 4$.

(b) $f'(x) > 0$ for $x < 3$, $f'(3) = 0$, and $f'(x) < 0$ for $x > 3$.

Solution: Reading the Signs

Mark $(3, 4)$. To the left of $x = 3$ the graph rises; at $x = 3$ the tangent is horizontal; to the right the graph falls.



x	$x < 3$	$x = 3$	$x > 3$
$f'(x)$	$+$	0	$-$
$f(x)$	Increasing	—	Decreasing

Thus $(3, 4)$ is a relative maximum point. The conditions do not determine a unique function.

3 Second Derivative Rule

The second derivative describes how the **slope itself changes**.

Second Derivative Rule

On an interval where f is twice differentiable:

- If $f''(x) > 0$ throughout the interval, the function is **concave up**.
- If $f''(x) < 0$ throughout the interval, the function is **concave down**.

Concave up	Slopes increase
Concave down	Slopes decrease

Example 2: Sketching from Points, Slopes, and Concavity

Sketch a function $f(x)$ with:

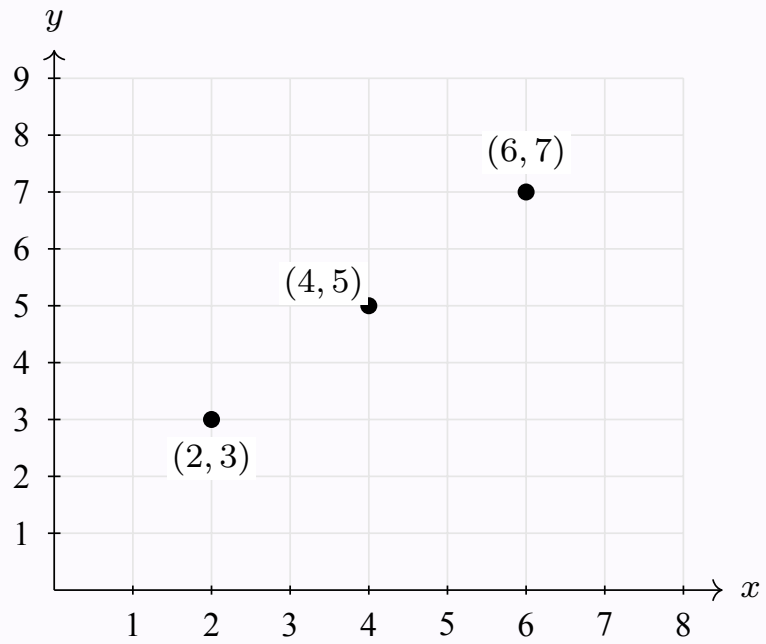
(a) $(2, 3)$, $(4, 5)$ and $(6, 7)$ on the graph.

(b) $f'(2) = 0$ and $f'(6) = 0$.

(c) $f''(x) > 0$ for $x < 4$, $f''(4) = 0$, and $f''(x) < 0$ for $x > 4$.

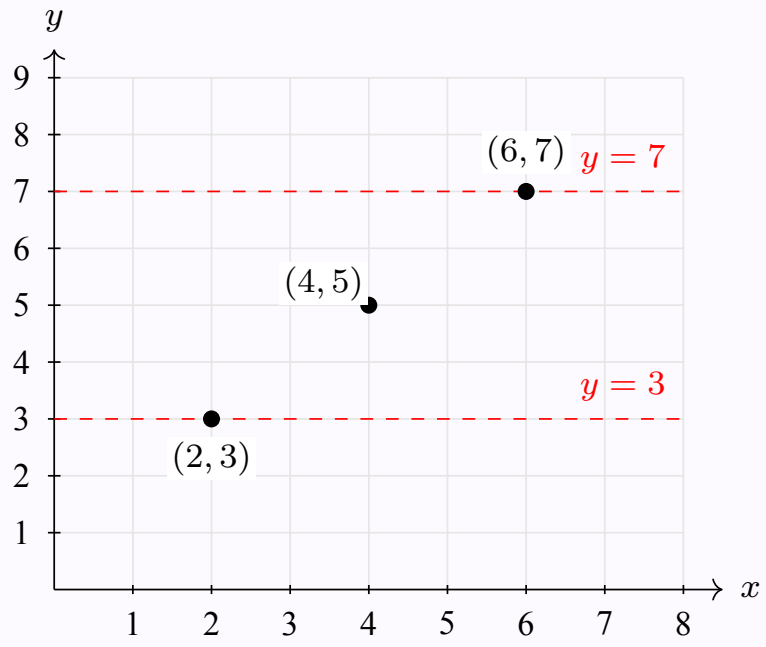
Step 1: Plot the Given Points

Mark the points $(2, 3),\ (4, 5),\ (6, 7)$.



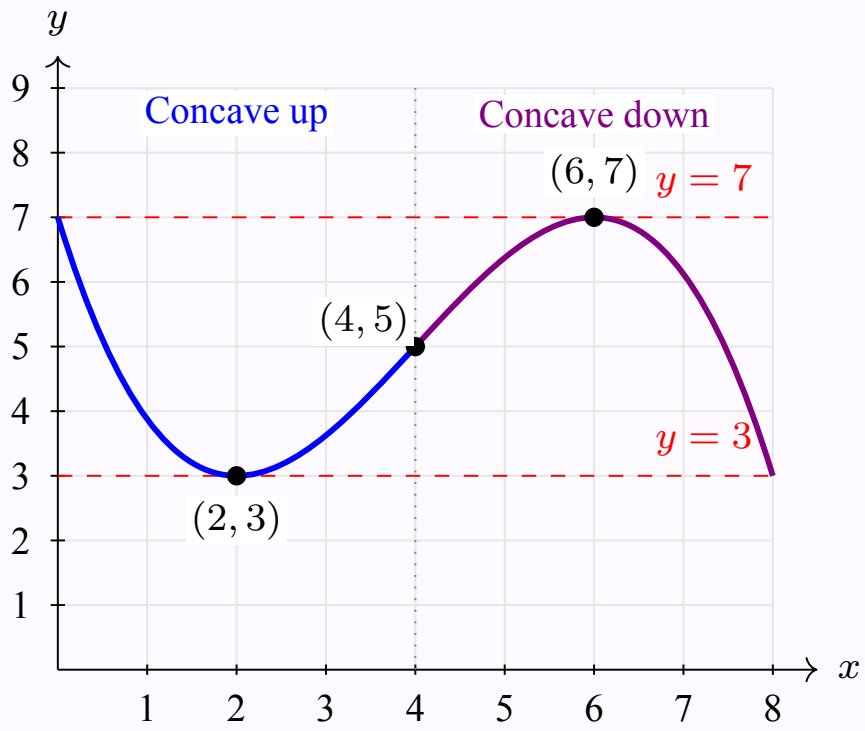
Step 2: Draw the Horizontal Tangents

Because $f'(2) = f'(6) = 0$, the tangent lines at $(2, 3)$ and $(6, 7)$ are horizontal.



Step 3: Concavity and the Complete Sketch

Since $f''(x) > 0$ for $x < 4$, the graph is concave up to the left of $x = 4$. So the graph looks like a **valley** before $x = 4$.
Since $f''(x) < 0$ for $x > 4$, it is concave down to the right. Therefore, the graph looks like a **hill** after $x = 4$.

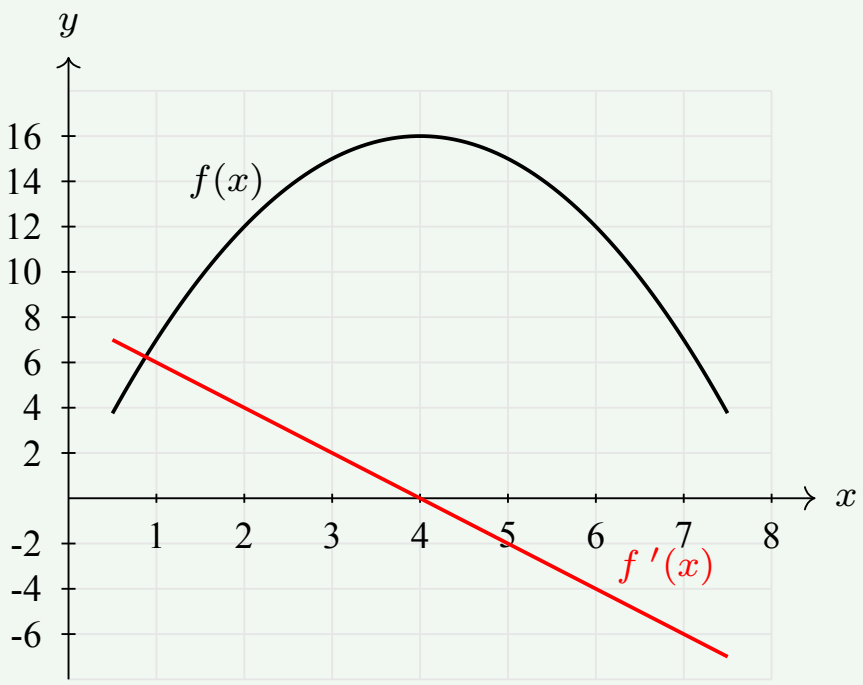


x	$x < 4$	$x = 4$	$x > 4$
$f''(x)$	$+$	0	$-$
Concavity	Up	Changes	Down

4 Connections Between a Function and Its Derivative

Example 3: Graphs of f and f'

The graph below shows a function f (black) and its derivative f' (red).



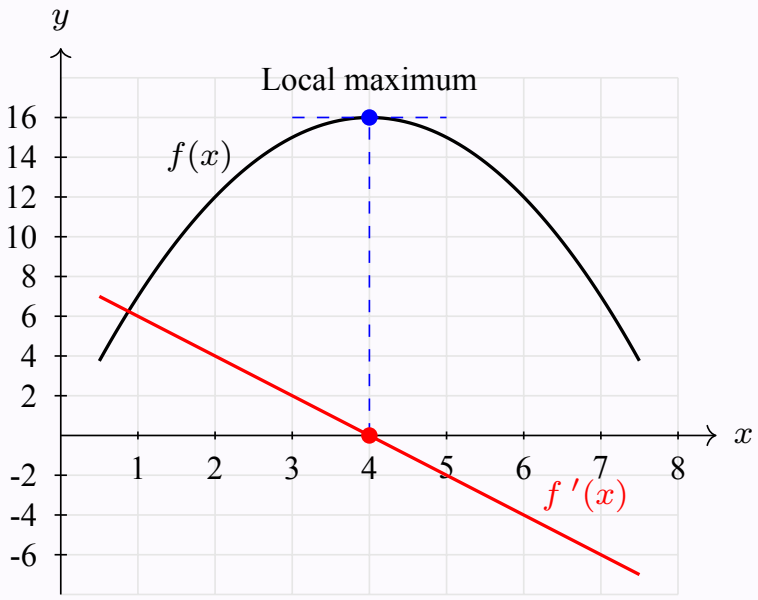
(a) Where does $f(x)$ have a local maximum?

(b) What is its slope at $x = 1$?

(c) What does the decreasing graph of $f'(x)$ tell us?

Solution (a): Local Maximum

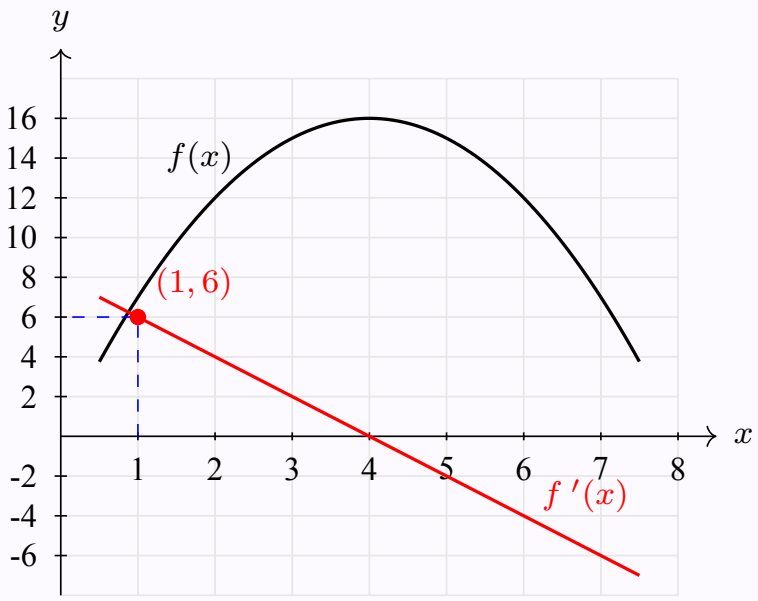
The function changes from increasing to decreasing at $x = 4$. Equivalently, the red derivative graph crosses 0 at $x = 4$.



f has a local maximum at $x = 4$.

Solution (b): Slope at $x = 1$

Read the height of the *red derivative graph* at $x = 1$:



$f'(1) = 6$.

Therefore, the slope of the tangent to f at $x = 1$ is 6.

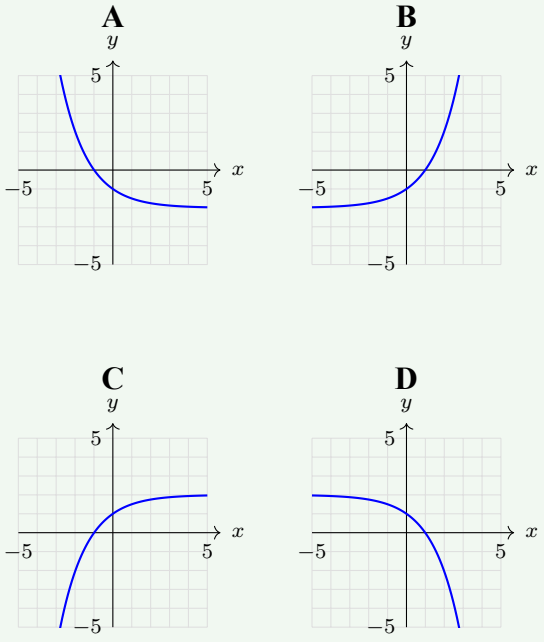
Solution (c): Concavity

The slope of $f(x)$ at a point a is just $f'(a)$. Since, the graph of $f'(x)$ decreases, then the slope of $f(x)$ decreases, hence $f(x)$ is concave down.
Alternatively, since $f'(x)$ decreases, by the first derivative rule its derivative $f''(x) = (f'(x))'$ is negative. By the second derivative rule, $f(x)$ is concave down.

5 Recognizing Behavior and Concavity

Example 4: Which Graph?

Which graph satisfies both properties: the **function** and the **slope** decrease as x increases?



Solution: Distinguishing Two Properties

The function must decrease (candidates A or D).
Its **slope** must also decrease, so it must be **concave down**. So the graph has to look like a **hill!**
Graph A is concave up, whereas D is concave down.

Correct choice: D.