Lecture 8 – Marginal Cost and Describing Graphs

Contents

1 Marginal Cost **2**

2 Describing the Graph of a Function **3**

2.1 Increasing and Decreasing Functions 3

2.2 Relative and Absolute Extrema 4

3 Concavity **6**

3.1 Inflection Points 6

3.2 Understanding Concavity: Valley or Hill ? 7

4 Intercepts and Asymptotes **8**

5 Strategy for Describing a Graph **10**

Section Goals

In this lecture, we will review marginal cost and learn to describe a graph using:

- increasing and decreasing intervals;
- relative and absolute maxima and minima;
- concavity and inflection points;
- intercepts, undefined points, and asymptotes.

1 Marginal Cost

Marginal Cost

Recall that if $C(x)$ is the cost of producing x units, then $C'(x)$ is called the **marginal cost function**. The marginal cost can be used to approximate the additional cost associated with a small increase in production:

$$C(b) - C(a) \approx C'(a)(b - a).$$

Example 5: Marginal Cost

Suppose the cost, in thousands of dollars, of producing x units is

$$C(x) = 6x^2 + 2x + 10.$$

Find:

(a) the marginal cost function;

(b) the cost and marginal cost when 10 units are produced;

(c) the estimated cost of producing the 11th unit;

(d) the estimated additional cost from 10 to 10.5 units.

Solution (a): Marginal cost function

The marginal cost function is the derivative $C'(x)$. Since

$$C(x) = 6x^2 + 2x + 10,$$

we get

$$C'(x) = (6x^2 + 2x + 10)' = 12x + 2$$

Solution (b): Cost and marginal cost at $x = 10$

First, calculate the cost when 10 units are produced:

$$C(10) = 6(10)^2 + 2(10) + 10 = 600 + 20 + 10 = 630.$$

Now calculate the marginal cost:

$$C'(10) = 12(10) + 2 = 122.$$

Solution (c): Estimated cost of the 11th unit

Be careful! We are not looking for $C(11)$, which represents the total cost of producing 11 units. To find the cost of producing the 11th unit, we need to calculate the cost of producing 11 units and subtract the cost of producing 10 units. Therefore, the difference $C(11) - C(10)$ represents the cost of producing the 11th unit!
Use

$$C(b) - C(a) \approx C'(a)(b - a).$$

Here, $a = 10$ and $b = 11$, so $b - a = 1$. Therefore,

$$C(11) - C(10) \approx C'(10)(11 - 1) = 122(11 - 10) = 122$$

The estimated cost of producing the 11th unit is 122 thousand dollars.

Solution (d): Estimated additional cost from 10 to 10.5 units

Follow faithfully the solution of part (c). The correct answer is 61.

2 Describing the Graph of a Function

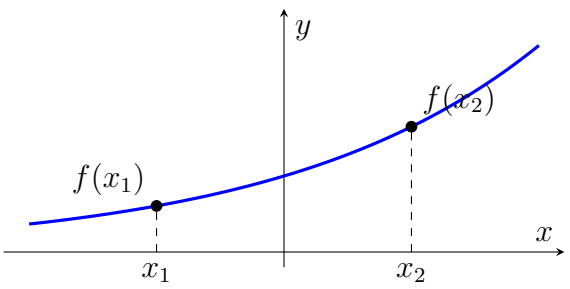
Today we learn how to describe the graph of a function. We focus on increasing and decreasing behavior, relative and absolute extrema, concavity, inflection points, intercepts, and asymptotes.

2.1 Increasing and Decreasing Functions

Increasing Functions

A function $f(x)$ is **increasing** on an interval if its graph rises from left to right. Mathematically, for x_1, x_2 in that interval,

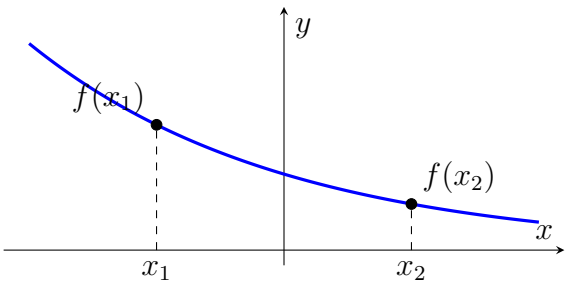
$$\boxed{x_1 < x_2 \implies f(x_1) < f(x_2).}$$



Decreasing Functions

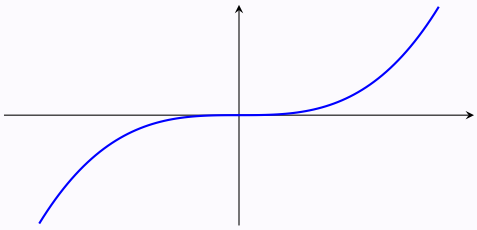
A function $f(x)$ is **decreasing** on an interval if its graph falls from left to right. Mathematically, for x_1, x_2 in that interval,

$$\boxed{x_1 < x_2 \implies f(x_1) > f(x_2).}$$



Quick Question

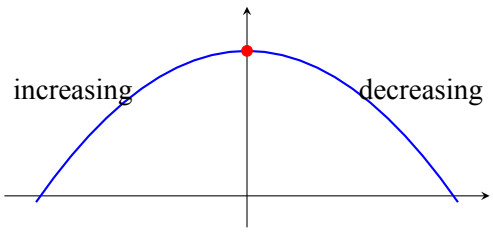
Is the function $f(x) = x^3$ increasing or decreasing?



2.2 Relative and Absolute Extrema

Relative Maximum

A **relative maximum** occurs when the graph changes from **increasing to decreasing**.
Think: *up* $\longrightarrow$ *down*.



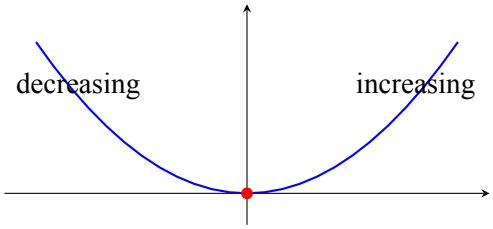
Absolute Maximum

An **absolute maximum** is the largest value $f(x)$ obtains on its **entire domain**.

A function can have a relative maximum without having an absolute maximum.

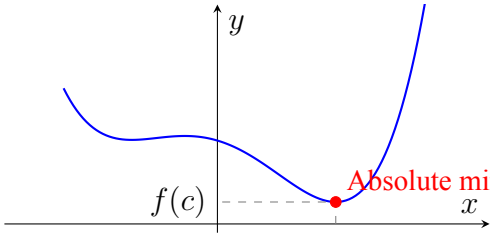
Relative Minimum

A **relative minimum** occurs when the graph changes from **decreasing to increasing**.
Think: *down* $\longrightarrow$ *up*.



Absolute Minimum

An **absolute minimum** is the smallest value $f(x)$ obtains on its **entire domain**.



Intuitive Summary

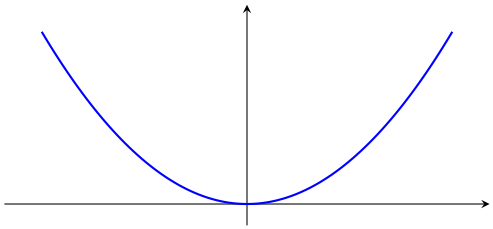
- A function is increasing (respectively, decreasing) if its graph goes only upwards (respectively, downwards) as we move from left to right.
- **Relative minima and maxima** describe a change in increasing/decreasing behavior:
  - **decreasing to increasing** indicates a **relative minimum**,
  - while **increasing to decreasing** indicates a **relative maximum.**
- **Absolute maxima and minima** are the greatest and smallest values of the function, respectively, over its entire domain.

3 Concavity

Concave Up

A **concave up** graph bends like the bottom of a **valley** or a **bowl**. As we move from left to right, its **slope increases**.

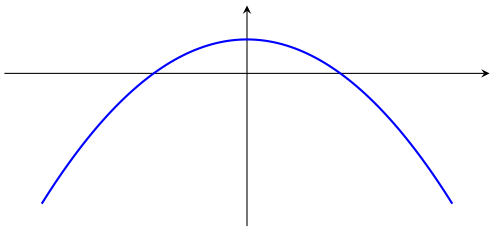
Think: Valley $\cup$



Concave Down

A **concave down** graph bends like the top of a **hill**. As we move from left to right, its **slope decreases**.

Think: Hill $\cap$

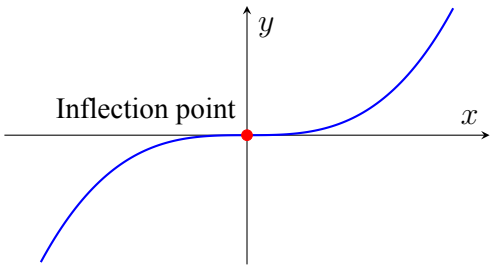


3.1 Inflection Points

Inflection Point

An **inflection point** is a point where the graph **changes concavity**:

concave up $\longleftrightarrow$ concave down.

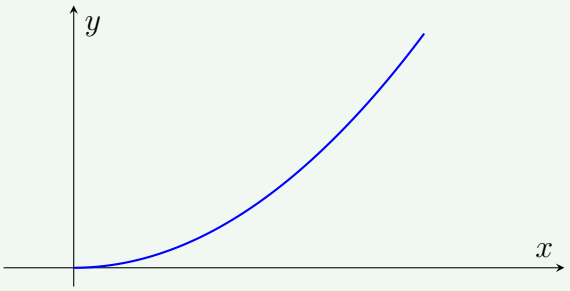


For $f(x) = x^3$, $(0, 0)$ is an inflection point.

3.2 Understanding Concavity: Valley or Hill ?

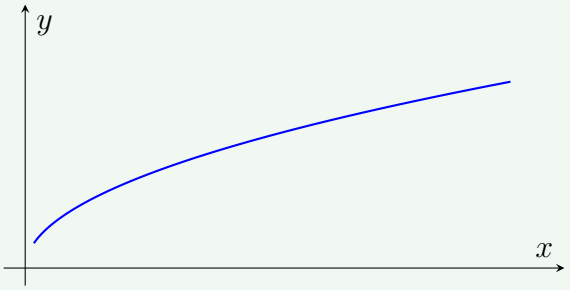
The following four graphs illustrate how concavity is related to the way the **slope changes** as we move from left to right. In particular, **increasing/decreasing** and **concave up/down** describe different properties.

Graph 1



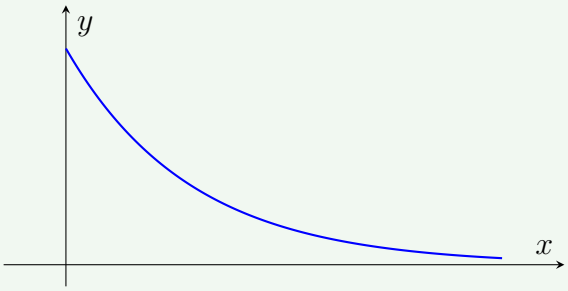
Concave up: The graph is increasing, and its slope is also increasing. **In simpler words, the graph looks like a valley !**

Graph 2



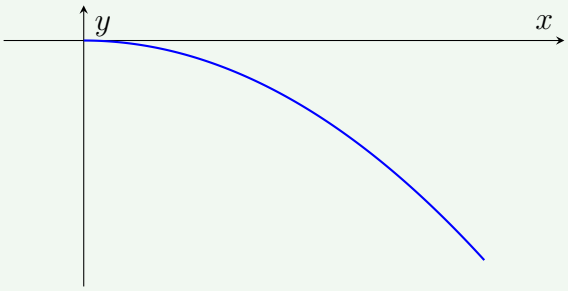
Concave down: The graph is increasing, but its slope is decreasing. **In simpler words, the graph looks like a hill !**

Graph 3



Concave up: The graph is decreasing, and its slope is also increasing. **In simpler words, the graph looks like a valley !**

Graph 4

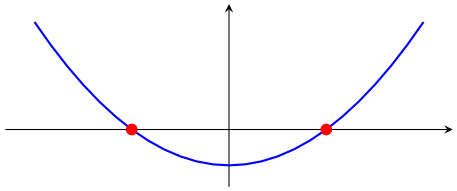


Concave down: The graph is decreasing, and its slope is decreasing. **In simpler words, the graph looks like a hill !**

4 Intercepts and Asymptotes

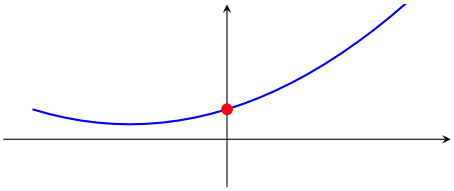
x-Intercepts

The **x-intercepts** are the points where the graph crosses or touches the x-axis. On the x-axis, $y = 0$.



y-Intercept

The **y-intercept** is the point where the graph crosses the y-axis. On the y-axis, $x = 0$.



Vertical Asymptotes

A **vertical asymptote** is a vertical line $x = a$ that the function approaches while its values become arbitrarily large in magnitude near a. For a rational function, possible vertical asymptotes occur where the denominator is zero (after canceling any common factors).

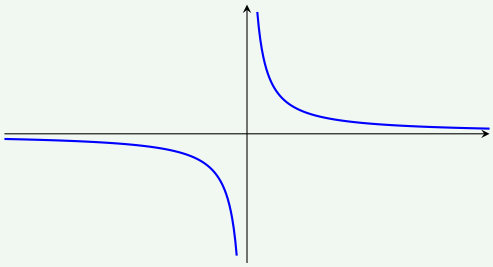
Horizontal Asymptotes

A **horizontal asymptote** is a horizontal line $y = L$ such that $f(x)$ approaches L as x becomes very large or very small. A graph may cross a horizontal asymptote.

Example: $f(x) = 1/x$

Consider the graph of

$$f(x) = \frac{1}{x}.$$



The vertical asymptote is $x = 0$ and the horizontal asymptote is $y = 0$.

5 Strategy for Describing a Graph

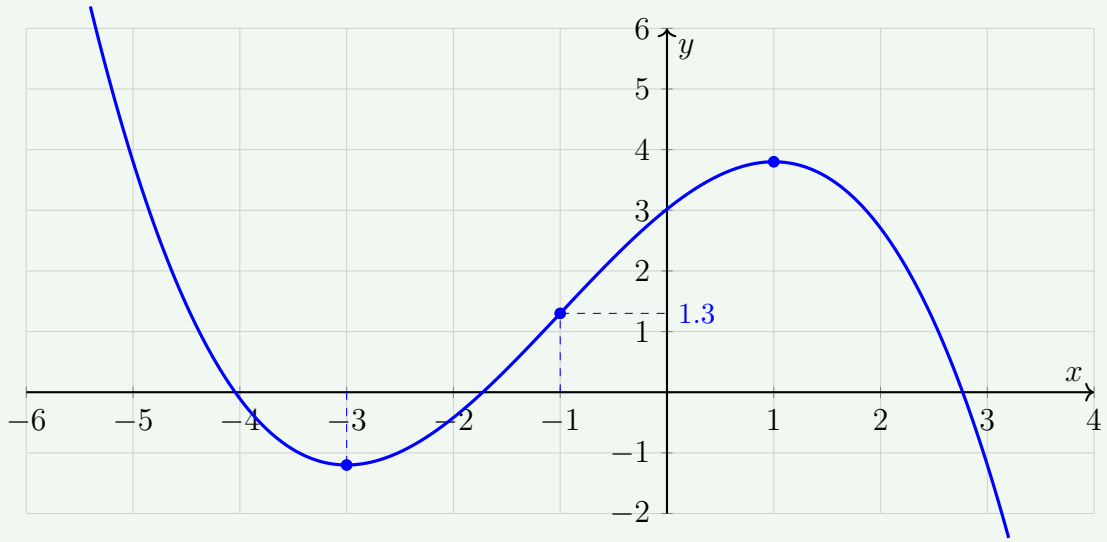
Graph-Description Checklist

When describing a graph, check:

1. Increasing/decreasing intervals.
2. Relative maxima and minima.
3. Absolute maximum and minimum.
4. Concavity.
5. Inflection points.
6. Intercepts.
7. Undefined points and asymptotes.

Example 1: Describing a Graph

Consider the following graph.



(a) On which interval(s) is the function increasing?

(b) On which interval(s) is the function decreasing?

(c) Where is/are the relative maximum point(s)?

(d) Where is/are the relative minimum point(s)?

(e) What is the absolute maximum value?

(f) What is the absolute minimum value?

(g) On which interval(s) is the function concave up or down?

(h) What is/are the inflection point(s)?

(i) What is/are the x-intercept(s)?

(j) What is/are the y-intercept(s)?

(k) What is/are the undefined point(s)?

(l) What is/are the asymptote(s)?

Solution (a)–(b): Increasing and decreasing

The graph rises from left to right from $x = -3$ to $x = 1$. Therefore, f is increasing on

$$\boxed{(-3, 1).}$$

The graph decreases before $x = -3$ and again after $x = 1$. Therefore, f is decreasing on

$$\boxed{(-\infty, -3) \quad \text{and} \quad (1, \infty).}$$

Solution (c)–(d): Relative extrema

The graph changes from increasing to decreasing at $x = 1$, so there is a relative maximum at

$$\boxed{x = 1 \quad (f(1) = 3.8).}$$

The graph changes from decreasing to increasing at $x = -3$, so there is a relative minimum at

$$\boxed{x = -3 \quad (f(-3) = -1.2).}$$

Solution (e)–(f): Absolute extrema

The graph continues upward on the left and downward on the right. Therefore,

$$\boxed{\text{No absolute maximum and no absolute minimum.}}$$

Solution (g)–(h): Concavity and inflection

From the graph, the line looks like a valley until $x = -1$, the function is **concave up** on

$$\boxed{(-\infty, -1).}$$

After $x = -1$, the graph looks like a hill. Therefore, the function is concave down on

$$\boxed{(-1, \infty).}$$

The graph changes concavity at $x = -1$. Since $f(-1) = 1.3$, the inflection point is

$$\boxed{(-1, 1.3).}$$

Solution (i)–(j): Intercepts

The graph has three x-intercepts, approximately

$$\boxed{(-4.044, 0), \quad (-1.725, 0), \quad (2.769, 0).}$$

The graph crosses the y-axis at $f(0) = 483/160 \approx 3.019$. Therefore, the y-intercept is approximately

$$\boxed{(0, 3.019).}$$

Solution (k)–(l): Undefined points and asymptotes

There are no holes, jumps, or breaks in the graph. Therefore, there are no undefined points. The graph is a polynomial and does not approach a vertical, horizontal, or oblique line as $x \to \pm\infty$. Therefore, there are no asymptotes.