Lecture 7 – The Derivative as a Rate of Change

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Section Goals

We will learn how to:

- calculate average and instantaneous rates of change;
- find velocity and acceleration from position;
- approximate changes in a function using its derivative;
- interpret the derivative in an applied setting.

1 Average and Instantaneous Rate of Change

Let $f(x)$ be a function defined on the interval $[a, b]$.

Average Rate of Change

$$\boxed{\frac{f(b) - f(a)}{b - a}}$$

This is the slope of the line segment from

$$(a, f(a)) \quad \text{to} \quad (b, f(b)).$$

Instantaneous Rate of Change

$$\boxed{f'(a)}$$

This is the slope of the tangent line at $x = a$.

Example 1

If

$$f(x) = x^2,$$

calculate:

(a) the average rate of change over $[2, 2.5]$;

(b) the average rate of change over $[2, 2.1]$;

(c) the instantaneous rate of change at $x = 2$.

Solution 1(a)

We use

$$\frac{f(b) - f(a)}{b - a}.$$

Here, $a = 2$ and $b = 2.5$. Since

$$f(2) = 4, \qquad f(2.5) = 6.25,$$

we obtain

$$\frac{f(2.5) - f(2)}{2.5 - 2} = \frac{6.25 - 4}{0.5} = \boxed{4.5}.$$

Solution 1(b)

We use the same formula, now with $a = 2$ and $b = 2.1$. Since

$$f(2) = 4, \qquad f(2.1) = 4.41,$$

we get

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{4.41 - 4}{0.1} = \boxed{4.1}.$$

Solution 1(c)

The **instantaneous rate of change** at $x = 2$ is $f'(2)$. Since

$$f(x) = x^2, \quad \text{and} \quad f'(x) = 2x,$$

we obtain

$$\boxed{f'(2) = 2(2) = 4}.$$

2 Velocity and Acceleration

Let $s(t)$ denote the position of an object moving in a straight line at time t.

Velocity

$$\boxed{v(t) = s'(t)}$$

Acceleration

$$\boxed{a(t) = v'(t) = s''(t)}$$

Example 2

A ball is shot straight up into the air. Its height, in feet, after t seconds is

$$s(t) = -16t^2 + 128t + 5.$$

Find:

(a) the velocity after 2 seconds;

(b) the acceleration after 2 seconds;

(c) when the velocity is -32 feet per second;

(d) when the ball is 117 feet above the ground.

Velocity and Acceleration

Since velocity is the derivative of position,

$$v(t) = s'(t).$$

Since $s(t) = -16t^2 + 128t + 5$, we get

$$v(t) = s'(t) = -32t + 128.$$

Also, acceleration is given by

$$a(t) = v'(t) = s''(t).$$

Therefore,

$$a(t) = (-32t + 128)' = (-32t)' + (128)' = -32.$$

Solution 2(a)–(b)

(a) Velocity after 2 seconds:

$$v(2) = -32(2) + 128 = -64 + 128 = \boxed{64 \text{ ft/sec}}.$$

(b) Since $a(t) = -32$ is a constant function,

$$\boxed{a(2) = -32 \text{ ft/sec}^2}.$$

Solution 2(c)

The word **when** indicates that we need to find the time t at which the velocity $v(t)$ equals -32. In other words, we need to solve the equation:

$$v(t) = -32.$$

Since $v(t) = -32t + 128$, we get

$$-32t + 128 = -32.$$

Subtract 128:

$$-32t = -160.$$

Divide by -32:

$$\boxed{t = 5 \text{ seconds}}.$$

Solution 2(d)

Again, the word **when** indicates that we need to find the time t at which the height $s(t)$ equals 117. In other words, we need to solve the equation:

$$s(t) = -117$$

Thus,

$$-16t^2 + 128t + 5 = 117.$$

Subtract 117:

$$-16t^2 + 128t - 112 = 0.$$

Divide by -16:

$$t^2 - 8t + 7 = 0.$$

Using the quadratic formula,

$$t = \frac{8 \pm \sqrt{8^2 - 4 \cdot 1 \cdot 7}}{2} = \frac{8 \pm \sqrt{36}}{2} = \frac{8 \pm 6}{2}.$$

So,

$$\boxed{t = 1 \quad \text{or} \quad t = 7 \text{ seconds}}.$$

3 Approximating the Change in a Function

We can approximate change in a function using the derivative. Suppose x changes from a to b. If b is close to a, then

Approximation Formula

$$f(b) - f(a) \approx f'(a)(b - a) \tag{1}$$

Equivalently,

$$f(b) \approx f(a) + f'(a)(b - a). \tag{2}$$

Very Important !

In approximation problems,

- a is the input where information is already known
- b is the input the problem asks about

Usually, the problem gives $f(a)$ and $f'(a)$, and asks us to estimate $f(b)$.

Example 3

Suppose that

$$f(500) = 7000 \qquad \text{and} \qquad f'(500) = 20.$$

Estimate $f(499.75)$.

Solution 3

Here,

$$a = 500 \quad \text{and} b = 499.75.$$

So,

$$b - a = 499.75 - 500 = -0.25.$$

Using the formula 2 :

$$f(b) \approx f(a) + f'(a)(b - a),$$

we get

$$\begin{aligned} f(499.5) &\approx f(500) + f'(500)(499.75 - 500) \\ &\approx 7000 + 20(-0.25) \\ &\approx 7000 - 5 = 6995. \end{aligned}$$

Therefore,

$$\boxed{f(499.75) \approx 6995}.$$

Example 4

Let $p(x)$ calculate the number of units that can be produced when x units of labor are employed. Suppose

$$p(5000) = 300, \qquad p'(5000) = 2.$$

(a) Interpret $p(5000) = 300$.

(b) Interpret $p'(5000) = 2$.

(c) Estimate $p(5000.5)$.

Solution 4(a)–(b)

a. $p(5000) = 300$ means:

5000 units of labor produce 300 units of goods.

b. $p'(5000) = 2$ means:

The instantaneous rate of change at $x = 5000$ is 2.

Solution 4(c)

We want to estimate $p(5000.5)$. We use the formula 2 :

$$p(b) \approx p(a) + p'(a)(b - a).$$

Here,

$$a = 5000 \quad \text{and} \quad b = 5000.5,$$

We get

$$\begin{aligned} p(5000.5) &\approx p(5000) + p'(5000)(5000.5 - 5000) \\ &\approx 300 + 2(0.5) \\ &\approx 301 \end{aligned}$$

Therefore,

$$\boxed{p(5000.5) \approx 301}.$$

4 Summary

- Average rate of change : $\dfrac{f(b) - f(a)}{b - a}$
- Instantaneous rate of change at a : $f'(a)$
- velocity : $v(t) = s'(t)$, acceleration: $a(t) = v'(t) = s''(t)$,
- $f(b) \approx f(a) + f'(a)(b - a)$

5 Homework Practice – Lecture 7

Exercise 1. Suppose that $f(300) = 4000$ and $f'(300) = 30$. Estimate each of the following.

(a) $f(301)$

(b) $f(300.5)$

(c) $f(299)$

(d) $f(298)$

(e) $f(299.75)$

Exercise 2. A toy rocket fired straight up into the air has height $s(t) = 176t - 16t^2$ feet after t seconds.

(a) What is the rocket's initial velocity (when $t = 0$)?

(b) What is the velocity after 3 seconds?

(c) What is the acceleration when $t = 7$?

(d) At what time will the rocket hit the ground?

(e) At what velocity will the rocket be traveling just as it smashes into the ground?