Lecture 6 – More about Derivatives and Marginal Quantities

Contents

1 Evaluating Derivatives 1

2 Derivative as a Rate of Change 3

3 Marginal Cost 5

4 Marginal Revenue and Marginal Profit 6

5 Summary 8

6 Homework Practice – Lecture 6 8

Section Goals

We will learn how to:

- evaluate first and second derivatives at a given point;
- interpret derivatives as rates of change and use them for estimates;
- calculate marginal cost, marginal revenue, and marginal profit.

1 Evaluating Derivatives

Let a be a specific value of x. The value of $f(x)$ at $x = a$ is $f(a)$.

Evaluating Derivatives at a Point

The value of $f'(x)$ at $x = a$ is

$$f'(a) \quad \text{or} \quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}.$$

The value of $f''(x)$ at $x = a$ is

$$f''(a) \quad \text{or} \quad \left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=a}.$$

Strategy

1. If you want to find $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}$, find the **first** derivative and evaluate it at $x = a$.
2. If you want to find $\left.\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=a}$, find the **second** derivative and evaluate it at $x = a$.

Example 3

Let

$$s = t^3 - 2t^2 + 3t.$$

Find

$$\left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=-2} \quad \text{and} \quad \left.\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\right|_{t=-2}.$$

Solution 3

First derivative:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = (t^3)' - (2t^2)' + (3t)' = 3t^2 - 4t + 3.$$

Evaluate:

$$\left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=-2} = 3(-2)^2 - 4(-2) + 3 = 23.$$

Second derivative:

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}(3t^2 - 4t + 3) = 6t - 4.$$

Evaluate:

$$\left.\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\right|_{t=-2} = 6(-2) - 4 = -16.$$

Therefore,

$$\boxed{\left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=-2} = 23} \quad \text{and} \quad \boxed{\left.\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\right|_{t=-2} = -16.}$$

2 Derivative as a Rate of Change

The derivative measures how fast a function is changing.

Derivative as a Rate of Change

$$\boxed{f'(a) = \textbf{rate of change of } f(x) \textbf{ at } x = a.}$$

For an increase of 1 unit, we use the approximation

$$\boxed{f(a+1) - f(a) \approx f'(a).} \tag{1}$$

where if you send $f(a)$ to the right hand side we obtain

$$\boxed{f(a+1) \approx f(a) + f'(a).} \tag{2}$$

Important ! The word **rate** of change is an alternative way to say the word derivative !

Example 4: Department Store Sales

A department store has strong holiday sales but expects sales to decline rapidly in January. Specifically, it is estimated that on the x-th day of January, sales will be

$$S(x) = 3 + \frac{9}{(x+1)^2}$$

in thousands of dollars.

(a) Calculate $S(2)$ and $S'(2)$, and interpret those results.

(b) Use the results above to estimate $S(3)$ and compare this estimate with the actual value of $S(3)$.

Solution 4(a)

First calculate $S(2)$:

$$S(2) = 3 + \frac{9}{(2+1)^2} = 3 + \frac{9}{9} = 4.$$

Therefore,

$$\boxed{S(2) = 4.}$$

Now compute $S'(x)$. Rewrite

$$S(x) = 3 + 9(x+1)^{-2}.$$

From the **Sum Rule**,

$$S'(x) = (3)' + \left[9(x+1)^{-2}\right]' = \left[9(x+1)^{-2}\right]'.$$

From the **Constant Multiple Rule**,

$$S'(x) = 9\left[(x+1)^{-2}\right]'.$$

From the **General Power Rule** for $g(x) = x + 1$,

$$S'(x) = 9[-2(x+1)^{-3}] \cdot (x+1)' = -18(x+1)^{-3} = -\frac{18}{(x+1)^3}.$$

Now evaluate at $x = 2$:

$$S'(2) = -\frac{18}{(2+1)^3} = -\frac{18}{27} = -\frac{2}{3}.$$

Therefore,

$$\boxed{S'(2) = -\frac{2}{3}.}$$

Interpretation of the results. Since $S(2) = 4$ and $S'(2) = -\frac{2}{3}$, sales on the second day of January are estimated to be

$$\boxed{\$4000},$$

and they are decreasing at a rate of approximately

$$\boxed{\frac{2}{3} \text{ thousand dollars per day.}}$$

Solution 4(b)

To calculate the actual value $S(3)$, substitute $x = 3$:

$$S(3) = 3 + \frac{9}{(3+1)^2} = 3 + \frac{9}{16} = 3.5625.$$

When you see the word **estimate** use the formula 2. Using the derivative by 2, we estimate

$$S(3) \approx S(2) + S'(2) = 4 - \frac{2}{3} = \frac{10}{3} = 3.3333\ldots.$$

This approximation is close to the actual value, but it is not exact.

3 Marginal Cost

If $C(x)$ is a cost function, then the **marginal cost function** is

$$\boxed{C'(x)}$$

The cost for producing a–products is $C(a)$. Likewise, the cost for producing $a+1$–products is $C(a+1)$. The extra cost of increasing production from $x = a$ to $x = a+1$ is

$$C(a+1) - C(a).$$

Using the formula 1 we can estimate this difference as follows :

Marginal Cost

$$\boxed{C'(a) \approx C(a+1) - C(a).}$$

So the marginal cost estimates the cost of producing one more item.

Example 5: Cost Function

Suppose

$$C(x) = (0.005)x^3 - (0.5)x^2 + 28x + 300$$

is the cost of producing x items. The factory currently produces $x = 50$ items per day.

(a) What is the extra cost from 50 to 51?

(b) What is the marginal cost at $x = 50$?

Solution 5(a): Exact Extra Cost

The exact extra cost is $C(51) - C(50)$. We compute:

$$C(51) = 1090.755 \qquad \text{and} \qquad C(50) = 1075.$$

Therefore,

$$C(51) - C(50) = 1090.755 - 1075 = 15.755.$$

Solution 5(b): Marginal Cost

The marginal cost is $C'(50)$. First compute the derivative of $C(x)$:

$$C'(x) = 0.015x^2 - x + 28.$$

Now evaluate at $x = 50$:

$$C'(50) = 0.015(50)^2 - 50 + 28 = 37.5 - 50 + 28 = 15.5.$$

Exact Cost vs. Marginal Cost

Exact extra cost:

$$C(51) - C(50) = 15.755.$$

Marginal cost approximation:

$$C'(50) = 15.5 \approx C(51) - C(50).$$

The marginal cost is an approximation of the cost of producing one more item.

4 Marginal Revenue and Marginal Profit

- $R(x)$: revenue from producing x units.
- $C(x)$: cost of producing x units.
- $P(x)$: profit from producing x units.

Profit is revenue minus cost:

$$\boxed{P(x) = R(x) - C(x).}$$

Marginal Quantities

The word marginal means **derivative !**

- **Marginal cost:** $C'(x)$
- **Marginal revenue:** $R'(x)$
- **Marginal profit:** $P'(x)$

Example 6: Revenue Function

Suppose revenue is

$$R(x) = 9x - 0.02x^2.$$

(a) Find the marginal revenue at $x = 30$.

(b) Find the production levels where revenue is \$400.

Solution 6(a): Marginal Revenue

The marginal revenue is $R'(x)$. Since

$$R(x) = 9x - 0.02x^2,$$

we get

$$R'(x) = 9 - 0.04x.$$

Now evaluate at $x = 30$:

$$R'(30) = 9 - 0.04(30) = 9 - 1.2 = 7.8.$$

Solution 6(b): Revenue Equals \$400

The production level is denoted by x. Hence, we are looking the x values where the revenue $R(x)$ equals 400. In other words, we solve the equation :

$$R(x) = 400.$$

Since $R(x) = 9x - 0.02x^2$, we get

$$9x - 0.02x^2 = 400.$$

Move everything to one side:

$$-0.02x^2 + 9x - 400 = 0.$$

Multiply by -100:

$$2x^2 - 900x + 40000 = 0.$$

Divide by 2:

$$x^2 - 450x + 20000 = 0.$$

Using the quadratic formula,

$$x = \frac{450 \pm \sqrt{450^2 - 4(1)(20000)}}{2} = \frac{450 \pm \sqrt{122500}}{2}.$$

Since $\sqrt{122500} = 350$, we get

$$x = \frac{450 \pm 350}{2}.$$

Therefore,

$$\boxed{x = 50 \quad \text{or} \quad x = 400.}$$

5 Summary

Key Formulas

1. $f'(a) \approx f(a+1) - f(a)$
2. **Cost function:** $C(x)$, **Revenue function:** $R(x)$, **Profit function:** $P(x)$
3. $C'(a) \approx C(a+1) - C(a)$
4. **Marginal Cost:** $C'(x)$, **Marginal Revenue:** $R'(x)$, **Marginal Profit:** $P'(x)$
5. **Profit:** $P(x) = R(x) - C(x)$

6 Homework Practice – Lecture 6

Exercise 1 (Homework 2, Problem 14)**.** Let $C(x)$ be the cost (in dollars) of manufacturing x bicycles per day in a certain factory. Interpret $C(90) = 5000$ and $C'(90) = 65$.

What does $C(90) = 5000$ represent in the context of this problem?

A. The cost of producing 90 bicycles is \$5000.

B. The cost of producing the 91st bicycle is \$5000.

C. The marginal cost of producing 90 bicycles is \$5000.

What does $C'(90) = 65$ represent in the context of this problem?

A. The cost of producing 90 bicycles is \$65.

B. The approximate cost of producing any one bicycle is \$65.

C. The approximate cost of producing the 91st bicycle is \$65.

Exercise 2 (Homework 2, Problem 15)**.** Suppose the revenue from producing (and selling) x units of a product is given by $R(x) = 9x - .02x^2$ dollars.

(a) Find the marginal revenue at a production level of 15.

(b) Find the production levels where the revenue is \$700.