Lecture 5 – More about Derivatives

Contents

1 More about Derivatives **1**

1.1 Differentiating with Respect to Another Variable 1

2 Second Derivatives **4**

3 Evaluating Derivatives **6**

4 Derivative as a Rate of Change **7**

5 Homework Practice – Lecture 5 **7**

Section Goals

We will learn how to:

- differentiate with respect to variables other than x,
- find first and second derivatives,
- evaluate derivatives at specific values,
- interpret derivatives as rates of change.

1 More about Derivatives

1.1 Differentiating with Respect to Another Variable

Sometimes we use variables other than x and y.

For example, if time is the independent variable, then we may write

$$f(t) = t^2.$$

So the derivative notation changes:

Derivative Notation

$$\frac{\mathrm{d}y}{\mathrm{d}t}$$

means:

derivative of y with respect to t.

If s is a function of p, then

$$\frac{\mathrm{d}s}{\mathrm{d}p}$$

means the derivative of s with respect to p. Similarly, $\frac{\mathrm{d}y}{\mathrm{d}t}$ means the derivative of y with respect to t.

Differentiate with respect to the variable in the denominator.

Quick Example

Find the derivative $\dfrac{\mathrm{d}y}{\mathrm{d}t}$, where $y = t^2$.

By the power rule we get $y' = (t^2)' = 2t$.

Example 1(a)

Compute

$$\frac{\mathrm{d}s}{\mathrm{d}p} \quad \text{if} \quad s = 3(p^3 - 5p + 1)^7.$$

Solution 1(a)

We apply the general power rule

$$\frac{\mathrm{d}}{\mathrm{d}p}\left[g(p)^r\right] = r \cdot g(p)^{r-1} \cdot g'(p).$$

Let if $g(p) = p^3 - 5p + 1$, then $s = 3[g(p)]^7$. Using the General Power Rule,

$$\frac{\mathrm{d}s}{\mathrm{d}p} = 3 \cdot 7[g(p)]^{7-1} g'(p).$$

We have

$$g'(p) = 3p^2 - 5.$$

Therefore,

$$\frac{\mathrm{d}s}{\mathrm{d}p} = 21(p^3 - 5p + 1)^6(3p^2 - 5).$$

$$\boxed{\frac{\mathrm{d}s}{\mathrm{d}p} = 21(p^3 - 5p + 1)^6(3p^2 - 5)}$$

Example 1(b)

Compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(at^3 + xt^{-1} + 3s^2\right).$$

Solution 1(b)

Here a, x, s are treated as constants because we differentiate with respect to t.
By the Sum Rule we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(at^3 + xt^{-1} + 3s^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(at^3\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(xt^{-1}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(3s^2\right).$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(at^3) = 3at^2$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(xt^{-1}) = x(-1)t^{-2} = -xt^{-2},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(3s^2) = 0.$$

Finally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(at^3 + xt^{-1} + 3s^2\right) = 3at^2 - xt^{-2}.$$

Equivalently,

$$\boxed{3at^2 - \frac{x}{t^2}}.$$

Only t is the variable. Everything else is constant.

2 Second Derivatives

When we differentiate $f(x)$, we obtain

$$f'(x).$$

If we differentiate again, we obtain the **second derivative**:

$$f''(x).$$

Second Derivative

Alternative notation:

$$\boxed{y'', \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}, \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(f(x)\big)}.$$

Example 2(a)

Find the first and second derivatives of

$$y = 3x - 2.$$

Solution 2(a)

First derivative:

$$y' = (3x - 2)' = 3.$$

Second derivative:

$$y'' = (3)' = 0.$$

Therefore,

$$\boxed{y' = 3 \quad \text{and} \quad y'' = 0.}$$

Example 2(b)

Find the first and second derivatives of

$$f(x) = x^3 + \frac{1}{x}.$$

Solution 2(b)

Rewrite:

$$f(x) = x^3 + x^{-1}.$$

First derivative:

$$\boxed{f'(x) = 3x^2 - x^{-2}.}$$

Now differentiate again to find the second derivative:

$$\begin{aligned} f''(x) &= \left(3x^2 - x^{-2}\right)' \\ &= \left(3x^2\right)' - \left(x^{-2}\right)' \\ &= 6x + 2x^{-3}. \end{aligned}$$

Therefore,

$$\boxed{f''(x) = 6x + \frac{2}{x^3}.}$$

Example 2(c)

Find the first and second derivatives of

$$f(t) = \sqrt{1-t}.$$

Solution 2(c)

We see square root so we immediately rewrite:

$$f(t) = \sqrt{1-t} = (1-t)^{1/2}.$$

Using the General Power Rule for $g(t) = 1 - t$ and $r = 1/2$,

$$f'(t) = \frac{1}{2}(1-t)^{\frac{1}{2}-1}(1-t)'.$$

Since $(1-t)' = -1$,

$$\boxed{f'(t) = -\frac{1}{2}(1-t)^{-1/2}}$$

From

$$f'(t) = -\frac{1}{2}(1-t)^{-1/2},$$

differentiate again by applying the General Power Rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((1-t)^{-1/2}\right) = -\frac{1}{2}(1-t)^{-1/2-1} \cdot (1-t)' = \frac{1}{2}(1-t)^{-3/2}$$

Thus,

$$f''(t) = \frac{1}{2} \cdot \left[\frac{1}{2}(1-t)^{-3/2}\right] = \frac{1}{4}(1-t)^{-3/2}.$$

Hence,

$$\boxed{f''(t) = \frac{1}{t}(1-t)^{-3/2}}$$

3 Evaluating Derivatives

Let a be a specific value of x. The value of $f(x)$ at $x = a$ is

$$f(a).$$

Evaluating Derivatives at a Point

The value of $f'(x)$ at $x = a$ is

$$f'(a) \quad \text{or} \quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}.$$

The value of $f''(x)$ at $x = a$ is

$$f''(a) \quad \text{or} \quad \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=a}.$$

This is just a notation ! We just evaluate plug a either to the first or the second derivative regarding the context !

Example 3

Let

$$s = t^3 - 2t^2 + 3t.$$

Find

$$\left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=-2} \quad \text{and} \quad \left.\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\right|_{t=-2}.$$

Solution 3

First derivative:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 3t^2 - 4t + 3.$$

Evaluate at -2 :

$$\left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=-2} = 3(-2)^2 - 4(-2) + 3$$
$$= 23.$$

Second derivative:

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = 6t - 4.$$

Evaluate at -2 :

$$\left.\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\right|_{t=-2} = 6(-2) - 4$$
$$= -16.$$

Therefore,

$$\boxed{\left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=-2} = 23} \quad \text{and} \quad \boxed{\left.\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\right|_{t=-2} = -16.}$$

4 Derivative as a Rate of Change

The derivative measures how fast a function is changing.

Derivative as a Rate of Change

$$\boxed{f'(a) = \text{rate of change of } f(x) \text{ at } x = a.}$$

For an increase of 1 unit, we can use the approximation

$$\boxed{f(a+1) - f(a) \approx f'(a).}$$

So

$$\boxed{f(a+1) \approx f(a) + f'(a).}$$

5 Homework Practice – Lecture 5

Exercise 1 (Homework 1, Problem 17)**.** Find the derivative of $f(x) = x^7$ at $x = \frac{1}{2}$.

Exercise 2 (Homework 1, Problem 18)**.** Find the derivative of $f(x) = \frac{1}{x}$ at $x = \frac{5}{6}$.

Exercise 3 (Homework 1, Problem 19)**.** If $f(x) = x^4$, compute $f(-4)$ and $f'(-4)$.

Exercise 4 (Homework 2, Problem 9)**.** Find

$$\frac{\mathrm{d}}{\mathrm{d}h}\left(x^4h^3 + y^5h + z^2\right).$$

Exercise 5 (Homework 2, Problem 10)**.** Find the first and second derivatives.

$$y = \sqrt{x+1}.$$

Exercise 6 (Homework 2, Problem 11)**.** Find the first and second derivatives.

$$v = 2t^2 + 8t + 13.$$

Exercise 7 (Homework 2, Problem 12)**.** Compute the following.

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^2+6z+5)^6\Big|_{z=-1}.$$

Exercise 8 (Homework 2, Problem 13)**.** Compute the following.

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(2x^3 - x^2 + 9x - 3)\Big|_{x=2}.$$