Lecture 4 – Basic Differentiation Rules

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Section Goals

We will learn how to differentiate functions using:

- the Constant Multiple Rule,
- the Sum Rule,
- the Power Rule,
- the General Power Rule.

The main skill is to recognize which rule applies.

1 Basic Differentiation Rules

1.1 Constant Multiple Rule

Constant Multiple Rule

Suppose k is a constant and $f(x)$ is differentiable. Then

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[kf(x)\right] = k\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\right]}$$

In words: **Constants stay in front when we differentiate.**

For example,

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[5f(x)\big] = 5f'(x).$$

Reminder: Power Rule

Power Rule

For any real number r (where the expression is defined),

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(x^r) = rx^{r-1}.}$$

Examples:

- $\frac{\mathrm{d}}{\mathrm{d}x}(x^7) = 7x^{7-1} = 7x^6$
- $\frac{\mathrm{d}}{\mathrm{d}x}(x^{1/2}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2}.$

Quick Example: Constant Multiple Rule

Differentiate:

$$y = 4x^7.$$

Solution

Using the Constant Multiple Rule,

$$y' = 4\frac{\mathrm{d}}{\mathrm{d}x}(x^7).$$

Using the Power Rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^7) = 7x^{7-1} = 7x^6.$$

Therefore,

$$y' = 4\frac{\mathrm{d}}{\mathrm{d}x}(x^7) = 4(7x^6) = 28x^6.$$

Thus,

$$\boxed{y' = 28x^6.}$$

1.2 Sum Rule

Sum Rule

If $f(x)$ and $g(x)$ are differentiable, then

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\big[f(x)+g(x)\big]=\frac{\mathrm{d}}{\mathrm{d}x}\big[f(x)\big]+\frac{\mathrm{d}}{\mathrm{d}x}\big[g(x)\big].}$$

In words: **Differentiate each term separately.**

Quick Question

If $y=x+x^5$, what is y'?

Answer

$$\boxed{y'=(x)'+\left(x^5\right)'=1+5x^4.}$$

1.3 General Power Rule

General Power Rule

Suppose

$$y=[g(x)]^r.$$

Then

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left([g(x)]^r\right)=r[g(x)]^{r-1}g'(x).}$$

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First identify $g(x)$.

Quick Question: General Power Rule

Differentiate:

$$y=(x^2+1)^3.$$

Solution

Here,

$$g(x) = x^2 + 1.$$

Using the General Power Rule,

$$y' = 3(x^2+1)^{3-1}(x^2+1)'.$$

Since $(x^2+1)' = (x^2)' + (1)' = 2x$ we get :

$$y' = 3(x^2+1)^2(2x).$$

So

$$\boxed{y' = 6x(x^2+1)^2.}$$

2 Examples

Example (a)

Differentiate:

$$y = 4x^7.$$

Solution (a)

Rules used: **Constant Multiple Rule + Power Rule**.

$$y' = 4\left(x^7\right)' = 4(7x^{7-1}) = 28x^6$$

so

$$\boxed{y' = 28x^6.}$$

Example (b)

Differentiate:

$$y = 5\sqrt{x}.$$

Solution (b)

Rewrite first:

$$\sqrt{x} = x^{1/2}.$$

So

$$y = 5x^{1/2}.$$

Using the Constant Multiple Rule and Power Rule:

$$y' = \left(5x^{1/2}\right)' = 5\left(x^{1/2}\right)' = 5\left(\frac{1}{2}x^{\frac{1}{2}-1}\right).$$

So

$$\boxed{y' = \frac{5}{2}x^{-1/2}}.$$

Example (c)

Differentiate:

$$y = \frac{x^3}{3} - \frac{3}{x^3}.$$

Solution (c)

Rewrite using powers:

$$y = \frac{1}{3}x^3 - 3x^{-3}.$$

Apply the **Sum Rule**! Differentiate term by term:

$$\begin{aligned} y' &= \left(\frac{1}{3}\cdot x^3\right)' - \left(3x^{-3}\right)' \\ &= \frac{1}{3}\left(x^3\right)' - 3\left(x^{-3}\right)' \\ &= \frac{1}{3}\cdot\left(3x^{3-1}\right) - 3\left(-3\cdot x^{-3-1}\right) \\ &= x^2 + 9x^{-4} \end{aligned}$$

$$\boxed{y' = x^2 + 9x^{-4}}$$

Example (d)

Differentiate:

$$y = (x^3 + 2x^2 + 3)^4.$$

Solution (d)

This is a **General Power Rule** example.

$$g(x) = x^3 + 2x^2 + 3, \qquad g'(x) = 3x^2 + 4x.$$

Also $r = 4$. Therefore,

$$\begin{aligned} y' &= 4g(x)^{4-1}g'(x) \\ &= 4\left(x^3 + 2x^2 + 3\right)^3\left(3x^2 + 4x\right) \end{aligned}$$

So

$$\boxed{y' = 4\left(x^3 + 2x^2 + 3\right)^3\left(3x^2 + 4x\right)}$$

Example (e)

Differentiate:

$$y = 5\sqrt[3]{1 + x^4}.$$

Solution (e)

We will apply the **General Power Rule**. We notice

$$g(x) = 1 + x^4 \quad \text{and} \quad g'(x) = 4x^3.$$

Rewrite:

$$y = 5\sqrt[3]{1 + x^4} = 5\sqrt[3]{g(x)} = 5[g(x)]^{1/3}.$$

So $r = 1/3$ in this case.

Now $y = 5[g(x)]^{1/3}$, so by the **Constant Multiple Rule** and the **General Power Rule**:

$$\begin{aligned} y' &= 5\left([g(x)]^{1/3}\right)' \\ &= 5\left[\frac{1}{3}[g(x)]^{1/3-1}g'(x)\right] \\ &= 5\left[\frac{1}{3}(1+x^4)^{\frac{1}{3}-1}(4x^3)\right]. \end{aligned}$$

So

$$\boxed{y' = \frac{20}{3}x^3(1+x^4)^{-2/3}.}$$

Strategy for Examples (f)–(h)

For the last examples, the main idea is:

Rewrite first, then apply the General Power Rule.

Remember:

$$\frac{1}{g(x)} = [g(x)]^{-1} \qquad \text{and} \qquad \sqrt{x} = x^{1/2}.$$

We will focus mainly on:

- identifying $g(x)$,
- choosing the correct exponent,
- multiplying by $g'(x)$.

Example (f)

Differentiate:

$$y = \frac{1}{x^5 + 4x^2 - 3}.$$

Solution (f)

Rewrite:

$$y = (x^5 + 4x^2 - 3)^{-1}.$$

Here,

$$g(x) = x^5 + 4x^2 - 3, \qquad g'(x) = 5x^4 + 8x.$$

Using the General Power Rule,

$$\boxed{y' = -(x^5 + 4x^2 - 3)^{-2}(5x^4 + 8x).}$$

Example (g)

Differentiate:

$$y = x + \frac{3}{1 - 2x^2} + 8.$$

Solution (g)

Rewrite:

$$y = x + 3(1 - 2x^2)^{-1} + 8.$$

For the middle term,

$$g(x) = 1 - 2x^2, \qquad g'(x) = -4x.$$

So

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[3(1 - 2x^2)^{-1}\right] = 3\left[-(1 - 2x^2)^{-2}(-4x)\right].$$

Hence,

$$\boxed{y' = 1 + \frac{12x}{(1 - 2x^2)^2}.}$$

Example (h)

Differentiate:

$$y = \left(1 - \frac{\sqrt{x}}{6}\right)^{3/4}.$$

Solution (h)

Here,

$$g(x) = 1 - \frac{\sqrt{x}}{6} = 1 - \frac{1}{6}x^{1/2}.$$

$$g'(x) = -\frac{1}{6}\left(\frac{1}{2}x^{-1/2}\right) = -\frac{1}{12}x^{-1/2}.$$

Using the General Power Rule,

$$y' = \frac{3}{4}\left(1 - \frac{\sqrt{x}}{6}\right)^{-1/4}\left(-\frac{1}{12}x^{-1/2}\right).$$

3 Homework Practice – Lecture 4

Exercise 1. Differentiate:

$$f(x) = 3x^4 - 2x^2 + 5x - 7.$$

Exercise 2. Differentiate:

$$f(x) = \frac{4}{x^2} + 3\sqrt{x}.$$

Exercise 3. Differentiate:

$$f(x) = (2x^2 + 1)^3.$$

Exercise 4 (Homework 2, Problem 3)**.** Differentiate.

$$y = 20\sqrt{x}.$$

Exercise 5 (Homework 2, Problem 4)**.** Differentiate.

$$y = \frac{x}{4} - \frac{4}{x}.$$

Exercise 6 (Homework 2, Problem 5)**.** Differentiate.

$$y = (x^2 + 7x + 6)^3.$$

Exercise 7 (Homework 2, Problem 6)**.** Differentiate.

$$y = 16\sqrt[4]{4x^4 + 4}.$$

Exercise 8 (Homework 2, Problem 7)**.** Differentiate.

$$y = \frac{7}{1 - 2x^3}.$$

Exercise 9 (Homework 2, Problem 8)**.** Differentiate.

$$y = \left(\frac{\sqrt{x}}{4} + 9\right)^{3/2}.$$