Lecture 3 – The concept of derivative

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Section Goals

- Understand the derivative as the slope of a tangent line.
- Learn the power rule for functions of the form x^r.
- Find slopes of tangent lines at specific points.
- Find equations of tangent lines.

1 The Derivative

Motivation: Functions and Graphs

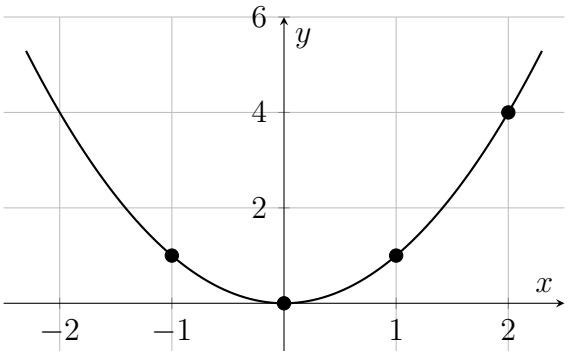
If $y = f(x)$ is a function, then the graph of f consists of all points of the form

$$(x, f(x)).$$

For example, if $f(x) = x^2$, then the graph contains

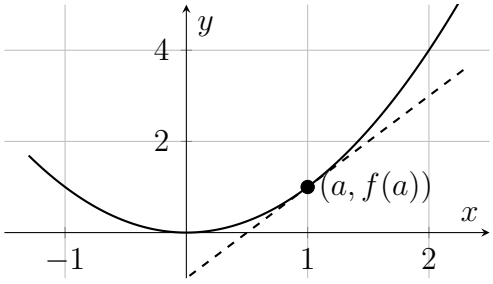
$$(0, 0), \qquad (1, 1), \qquad (2, 4), \qquad (-1, 1).$$

Example: The Graph of $f(x) = x^2$



Tangent Line at a Point

Choose a point on the graph corresponding to $x = a$. Then the point is $(a, f(a))$. At this point, we can draw the **tangent line** to the graph.



Derivative at a Point

Derivative at a point

The slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ is called the **derivative of f at $x = a$.**

We denote this number by

$$f\,'(a), \qquad \frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(f(x)), \qquad y\,'.$$

Thus,

$$f\,'(a) = \text{the slope of the tangent line to the graph of } f \text{ at } x = a.$$

Big Picture

$$\boxed{\text{function} \longrightarrow \text{graph} \longrightarrow \text{tangent line at } x = a \longrightarrow \text{slope} \longrightarrow f\,'(a)}$$

2 Derivatives of Known Functions

Derivatives of Some Known Functions

a. A linear function $f(x) = mx + b$ has derivative m, because its tangent line is itself and its slope is constant:

$$(mx + b)' = m.$$

So the derivative of $f(x)$ at any point is $f'(x) = m$.

b. A constant function $f(x) = b$ is also a line, namely $f(x) = 0x + b$, so

$$)' = 0.$$

So the derivative of $f(x)$ at any point is zero $f'(x) = 0$.

c. From the previous section, the slope of the tangent line to $f(x) = x^2$ is $2x$, so

$$\left(x^2\right)' = 2x.$$

So at any point the derivation of $f(x) = x^2$ is $f\,'(x) = 2x$.

Quick Examples

Find the derivative of the following functions :

a. $2x + 3$

b. $3x$

c. x

d. 8

Answers

a. $(2x + 3)' = 2$

b. $(3x)' = 3$

c. $(x)' = 1$

d. $(8)' = 0$

3 Power Rule

Power Rule

We know how to differentiate the first few powers of x:

$$x^0 = 1 \Rightarrow (1)' = 0, \qquad x^1 = x \Rightarrow (x)' = 1, \qquad (x^2)' = 2x.$$

What happens with the other powers of x ?

Power Rule

If

$$f(x) = x^r,$$

where r is any nonzero real number, then

$$\boxed{(x^r)' = rx^{r-1}.}$$

Quick Question

What is the derivative of x^6 ?

Answer

By applying the power rule $(x^r)' = rx^{r-1}$ for $r = 6$ we obtain :

$$\left(x^6\right)' = 6x^{6-1} = 6x^5.$$

Important Strategy

Before using the power rule, try to rewrite the function in the form

$$\boxed{x^r}$$

for some exponent r.

Original form	Rewrite as a power of x
x^3	x^3
$\dfrac{1}{x^3}$	x^{-3}
$\dfrac{1}{\sqrt[3]{x^5}}$	$x^{-5/3}$

Exponent Rules to Remember

i. $\dfrac{1}{x^n} = x^{-n}$

ii. $\sqrt[n]{x^m} = x^{m/n}$

Please make sure that you feel comfortable with the next example before you move to the following part of the lecture :

Example

Simplify the following expressions using the exponent rules:

i. $\dfrac{1}{x^3}$

ii. $\dfrac{1}{x^7}$

iii. $\sqrt{x}$

iv. $\sqrt[3]{x^2}$

v. $\dfrac{1}{\sqrt{x}}$

vi. $\dfrac{1}{\sqrt[4]{x^3}}$

vii. $\sqrt[3]{\dfrac{1}{x^5}}$

Answers

i. x^{-3}

ii. x^{-7}

iii. $x^{1/2}$

iv. $x^{2/3}$

v. $x^{-1/2}$

vi. $x^{-3/4}$

vii. $x^{-5/3}$

Example

Find the derivatives of the following functions:

(a) $f(x) = x^3$

(b) $f(x) = \dfrac{1}{x^3}$

(c) $f(x) = \dfrac{1}{\sqrt[3]{x^5}}$

Example 1 (a)

Recall the power rule:

$$\boxed{(x^r)' = rx^{r-1}.}$$

Here $r = 3$. Therefore,

$$f'(x) = 3x^{3-1} = 3x^2.$$

Hence,

$$\boxed{f\,'(x) = 3x^2.}$$

Example 1 (b)

First rewrite:

$$\frac{1}{x^3} = x^{-3}.$$

So $f(x) = x^{-3}$. Using the power rule,

$$f'(x) = -3x^{-3-1} = -3x^{-4}.$$

Therefore,

$$\boxed{f'(x) = -3x^{-4}}$$

Don't simplify more this expression if it is not needed !

Example 1 (c)

First rewrite:

$$\sqrt[3]{x^5} = x^{5/3}$$

so

$$f(x) = \frac{1}{x^{5/3}} = x^{-5/3}.$$

Using the power rule,

$$f'(x) = -\frac{5}{3}x^{-5/3-1}.$$

Since

$$-\frac{5}{3} - 1 = -\frac{8}{3},$$

we get

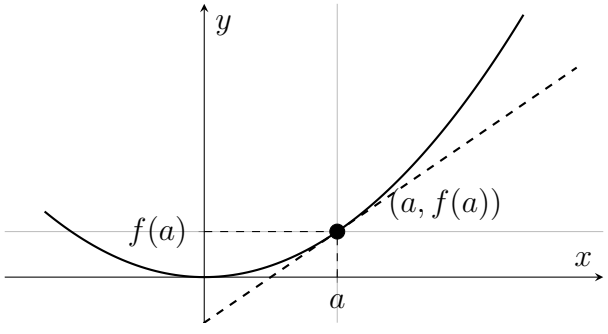
$$\boxed{f'(x) = -\frac{5}{3}x^{-8/3}}$$

Don't simplify more this expression if it is not needed !

4 Tangent Lines

So, let's go back to where we started. We defined the derivative of a function at a point as the slope of the corresponding tangent line at that point. Therefore, it is reasonable to ask: **What is the equation of the tangent line at $x = a$?** To answer this question, we will need the derivative as well as the point–slope formula! Look the following picture :

Geometric Picture



We observe that the point $(a, f(a))$ lies on both the tangent line and the graph of $f(x)$. Since we already know that the slope of the tangent line is $f'(a)$, we can apply the point–slope formula to obtain:

Equation of the tangent line: $y - f(a) = f'(a)(x - a)$.

How to find the tangent line

Given a function $y = f(x)$, the tangent line at $x = a$ satisfies the following formula :

$$y - f(a) = f'(a)(x - a)$$

Step 1: Identify the number a
Step 2: Find $f(a)$
Step 3: Find $f'(x)$
Step 4: Find $f'(a)$ **by subtituting to the previous formula.**
Step 5: Plug these numbers to the equation : $y - f(a) = f'(a)(x - a)$.

5 Examples

Example

Find the slope of $y = \dfrac{1}{x}$ at $x = 2$.
Then find the equation of the tangent line at this point and sketch both.

Find the slope

If $f(x) = \frac{1}{x}$, then the slope of $f(x)$ at $x = 2$ is

$$f'(2)$$

We find first the derivative ! Since $f(x) = x^{-1}$, by using the power rule :

$$(x^r)' = rx^{r-1}$$

for $r = -1$ we get :

$$f'(x) = \left(x^{-1}\right)' = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

At $x = 2$ we have that

$$\boxed{f'(2) = -\frac{1}{2^2} = -\frac{1}{4}}.$$

Find the equation of the tangent line

Since $a = 2$, the equation of the tangent line is :

$$y - f(2) = f'(2)(x - 2)$$

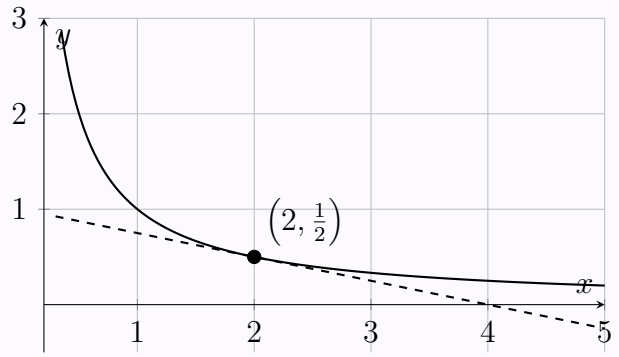
By the first part of the exercise we have that

$$f'(2) = -\frac{1}{4}$$

Now, $f(2) = \frac{1}{2}$ so

$$y - \frac{1}{2} = -\frac{1}{4}(x - 2)$$

Sketch



Example 3

Example

Find the equation of the tangent line to the graph of $f(x) = \sqrt{x}$ at $x = 1$.

Find the equation

We observe that $a = 1$. So we have to find the equation :

$$y - f(1) = f'(1)(x - 1)$$

We notice that $f(1) = \sqrt{1} = 1$. Now in order to find $f'(1)$ we find first the derivative $f'(x)$. Since

$$f(x) = \sqrt{x} = x^{1/2}$$

we apply the power rule $(x^r)' = rx^{r-1}$ for $r = 1/2$:

$$f'(x) = \left(x^{1/2}\right)' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$$

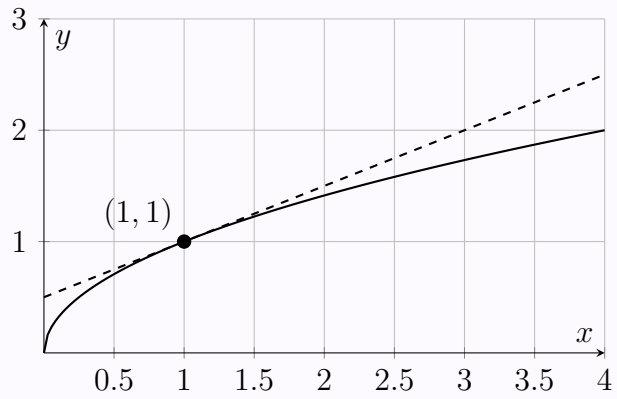
So

$$f'(1) = \frac{1}{2} \cdot 1^{-\frac{1}{2}} = \frac{1}{2}$$

Hence, the desired equation is

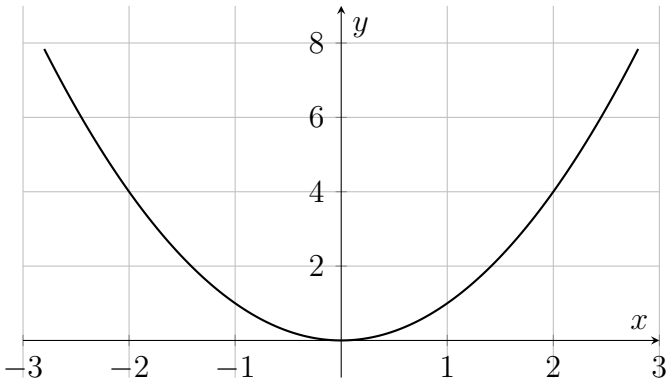
$$\boxed{y - 1 = \frac{1}{2}(x - 1)}$$

Sketch – Optional

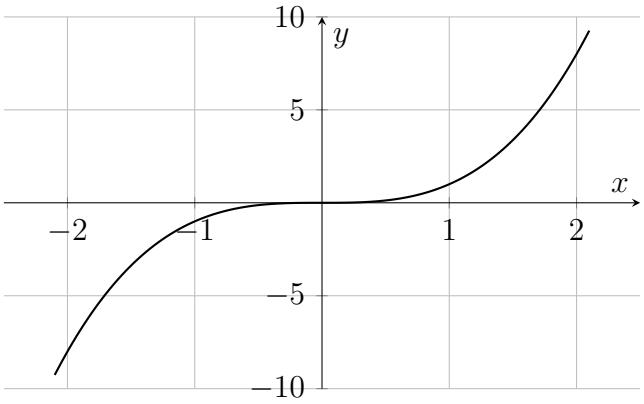


6 Graphs of important functions – Optional

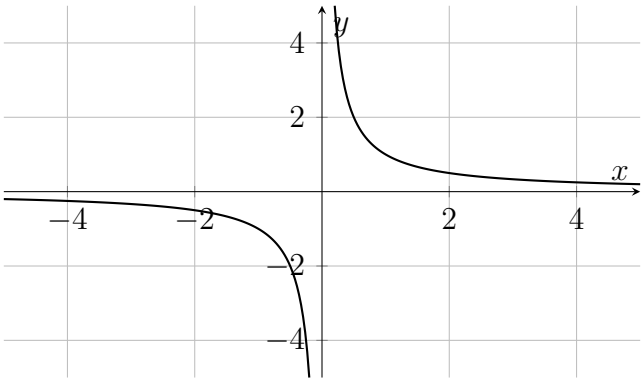
Graph of $y = x^2$



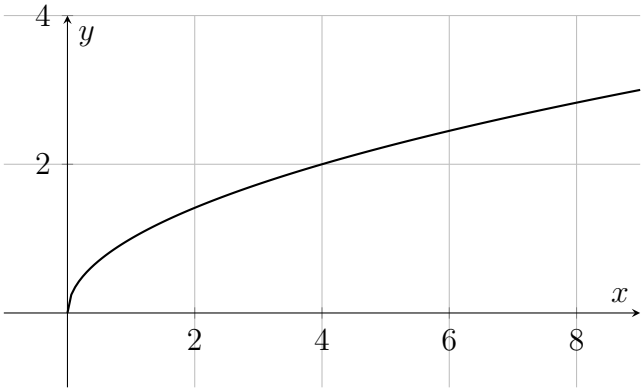
Graph of $y = x^3$



Graph of $y = \frac{1}{x}$



Graph of $y = \sqrt{x}$



Final Summary

- The derivative $f'(a)$ is the slope of the tangent line at $x = a$.
- The power rule says:
$$\boxed{(x^r)' = rx^{r-1}.}$$
- The tangent line at $x = a$ is:
$$\boxed{y - f(a) = f\,'(a)(x - a).}$$

7 Homework Practice – Lecture 3

Exercise 1 (Homework 1, Problem 10)**.** Write the equation of the tangent line to the graph of $y = x^2$ at the point $x = -1.9$.

Exercise 2 (Homework 1, Problem 11)**.** Find the derivative of $f(x) = x - 6$.

Exercise 3 (Homework 1, Problem 12)**.** Use the power rule to find the derivative of $f(x) = x^9$.

Exercise 4 (Homework 1, Problem 13)**.** Use the power rule to find the derivative of $f(x) = x^{-5}$.

Exercise 5 (Homework 1, Problem 14)**.** Use the power rule to find the derivative of $f(x) = x^{6/7}$.

Exercise 6 (Homework 1, Problem 15)**.** Use the power rule to find the derivative of $f(x) = \dfrac{1}{\sqrt{x^{17}}}$.

Exercise 7 (Homework 1, Problem 16)**.** Use the power rule to find the derivative of $f(x) = \sqrt[11]{x}$.

Exercise 8 (Homework 2, Problem 1)**.** Find the equation of the tangent line to the graph $y = f(x)$ at the given value of x.

$$f(x) = x^5, \quad x = 3.$$

What is the equation of the tangent line?

Exercise 9 (Homework 2, Problem 2)**.** Find an equation of the tangent line to the graph of $y = f(x)$ at the given x.

$$f(x) = \sqrt{x}, \quad x = \frac{1}{49}.$$

What is the equation of the tangent line?