Lecture 2 – The Slope of a Curve at a Point

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1 Reminder

a. **Slope-intercept form** $y = mx + b$, where m is the slope and b is the y-intercept.

b. **slope between two points:** If a line passes through

$$(x_1, y_1), \qquad (x_2, y_2),$$

then

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

c. **Point–slope formula:** If a line has slope m and passes through (x_1, y_1), then

$$\boxed{y - y_1 = m(x - x_1)}$$

2 Example 4 – Parallel and Perpendicular Lines

Let

$$L_1 : y = m_1 x + b_1 \qquad \text{and} \qquad L_2 : y = m_2 x + b_2$$

be two lines.

- L_1 and L_2 are **parallel** if they have the same slope:

$$\boxed{m_1 = m_2}.$$

- L_1 and L_2 are **perpendicular** if the product of their slopes is -1:

$$\boxed{m_1 m_2 = -1}.$$

Perpendicular Slopes

Equivalently, if $m_1 \neq 0$, then the slope of a perpendicular line is

$$\boxed{m_2 = -\frac{1}{m_1}.}$$

Quick Question

If a line has slope $m = 4$, what is the slope of:

1. a line parallel to it?
2. a line perpendicular to it?

Answer

Parallel slope: $\boxed{4}$. Perpendicular slope: $\boxed{-\frac{1}{4}}$.

Example

Find the equation of the line passing through $(5, 3)$ and parallel to the line

$$2x + 5y = 7.$$

Find the slope

First, we find the slope of the given line.Rewrite $2x + 5y = 7$ in slope-intercept form, so in other words solve for y :

$$2x + 5y = 7 \iff 5y = -2x + 7 \iff y = -\frac{2}{5}x + \frac{7}{5}.$$

Hence its slope is

$$m = -\frac{2}{5}.$$

Since parallel lines have the same slope, the line that we are looking for also has slope

$$m = -\frac{2}{5}.$$

Find the equation

Now use the point-slope formula:

$$y - y_1 = m(x - x_1).$$

Using the point $(5, 3)$, we obtain

$$y - 3 = -\frac{2}{5}(x - 5).$$

3 Example 5 – Vertical and Horizontal Lines

- A **vertical line** has equation

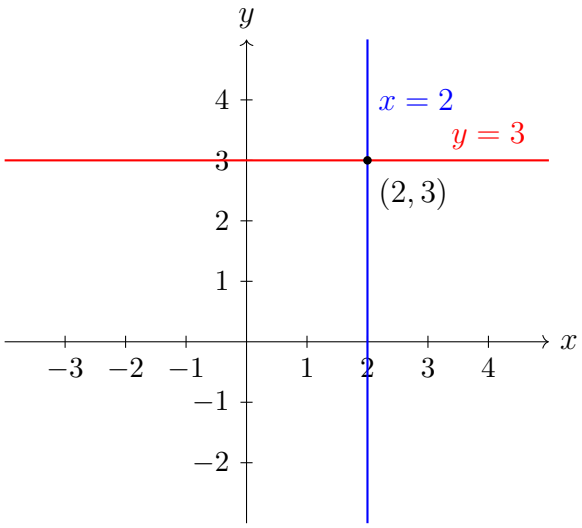
  $$\boxed{x = a}$$

  and has undefined slope.

- A **horizontal line** has equation

  $$\boxed{y = b}$$

  and has slope 0.



Quick Question

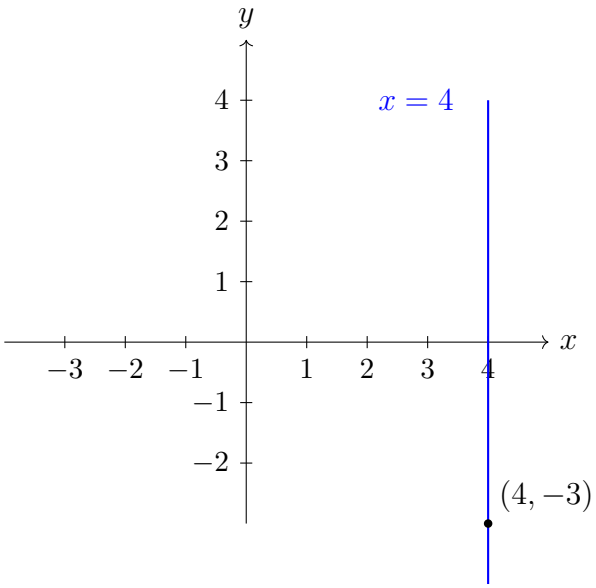
Quick Question

What is the equation of the vertical line passing through $(4, -3)$?

Answer

Since every point on this line has x-coordinate 4, the line's equation is

$$\boxed{x = 4.}$$



Example

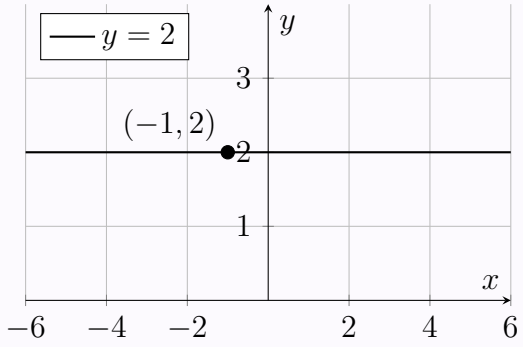
Write the equation of the horizontal line passing through $(-1, 2)$.

Find the equation

A horizontal line has the form $y = b$. Since the line passes through $(-1, 2)$, its y-coordinate is always equal to 2. Therefore,

$$\boxed{y = 2.}$$

Graphically:



4 Examples 6 and 7 – Applications of Calculus Terminology

Throughout this course, in addition to calculus, we will also learn useful terminology associated with the application of calculus to:

- business,
- finance,
- the biological sciences,
- the social sciences.

Example 6: Fixed and Marginal Cost

Fixed and Marginal Cost

Consider a manufacturer with cost function

$$C(x) = 2x + 1000,$$

where x is the number of units produced.
In this case, we will provide an economic interpretation of the slope and y-intercept.

Fixed Cost

When the manufacturer produces 0 units, the cost is

$$C(0) = \boxed{1000}.$$

We refer to the value of the function at $x = 0$ as the

$$\boxed{\text{fixed cost}}.$$

This might be the cost of paying rent, administrative staff, the utility bills on the facility, etc.

Marginal Cost

The slope, which is

$$m = \boxed{2},$$

is the cost of producing each additional item. We will refer to this as the

$$\boxed{\text{marginal cost}}.$$

Since our function is linear, the marginal cost is the same regardless of the number of units produced.

Example

When the owner of an outdoor company sets the price of one canoe at \$500, he can sell roughly 200 canoes per year.
If he sets the price at \$650, he can sell roughly 170 canoes per year.
Let $G(x)$ denote the number of canoes sold per year when the price is set at x dollars.
Assume that $G(x)$ is a linear function of x.

(a) Find a formula for $G(x)$.

(b) How many canoes will be sold if the price is set at \$600?

Decipher the Statement

We have a linear function $G(x)$, i.e. a **line's equation.**
Here x is the price of a single canoe, and $G(x)$ is how many canoes we are going to sell this year.

Two points

Since we are dealing with an equation of a line, we need either:

- the slope and a point, or
- two points.

From the information given, the graph of G passes through the two points

$$(500, 200) \qquad \text{and} \qquad (650, 170).$$

Find the slope

We first calculate the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{170 - 200}{650 - 500}.$$

Therefore,

$$m = \frac{-30}{150} = -\frac{1}{5}.$$

Find the formula

Now use the point-slope formula

$$y - y_1 = m(x - x_1).$$

Using the point $(500, 200)$, we obtain

$$G(x) - 200 = -\frac{1}{5}(x - 500).$$

Simplifying,

$$G(x) - 200 = -\frac{1}{5}x + 100,$$

and therefore

$$\boxed{G(x) = -\frac{1}{5}x + 300.}$$

How many canoes will be sold if the price is set at \$600?

We substitute $x = 600$ into our formula:

$$\boxed{G(x) = -\frac{1}{5}x + 300.}$$

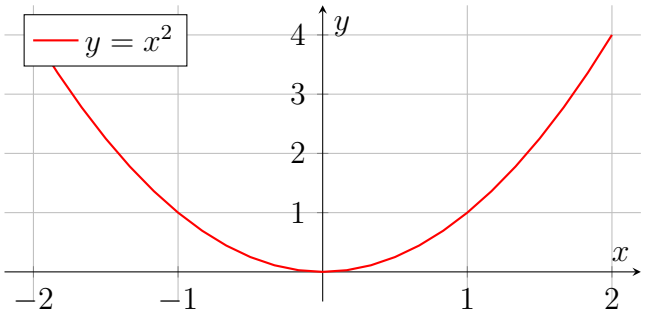
and we get

$$G(600) = -\frac{1}{5}(600) + 300 = -120 + 300 = 180$$

Therefore, 180 canoes will be sold per year when the price is set at \$600.

5 Slope of a curve – Example 1

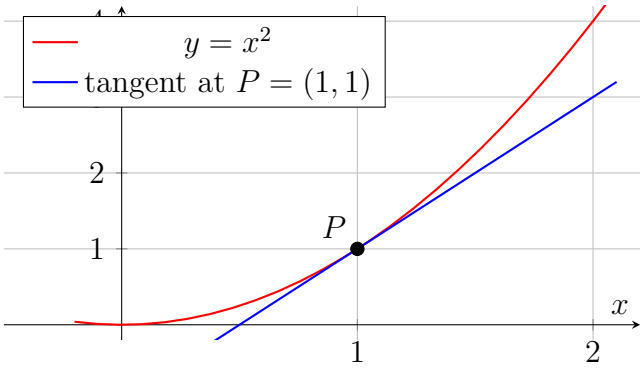
We have seen that the slope of a line is constant. For example, if I consider the line $y = 3x + 7$ then $m = 3$. This is not the case for the curves. One can see that point by point we have a different slope.



What do we desire ? We want to find the slope of a curve at a point P.

- First, we find the tangent line, which is the line that, near that point, touches the curve only once at P.
- Note that as we zoom in on the curve at that point, the curve begins to look more and more like a straight line.

The Tangent Line Near a Point



The tangent line gives the slope of the curve at the point.

Slope of a Curve

The **slope of a curve at point** P is defined to be the slope of the tangent line at P.

Fact: Slope of the line tangent to

$$y = x^2$$

is

$$\boxed{m = 2x}$$

at every point (x, y).

How to Use the Fact – Find the tangent line of $y = x^2$ at a point

For the curve $y = x^2$, the slope of the tangent line is $m = 2x$. So, to find the slope at a specific point $x = a$ (or at (a, a^2)):

1. Look at the x-coordinate of the point.

   x-coordinate $\longrightarrow$ **a**

2. Substitute it into

   $$m = 2x.$$

   so

   $$m = 2x \longrightarrow m = 2a \quad \textbf{at the point } \left(a, a^2\right)$$

3. Use the point-slope formula

$$y - y_1 = m(x - x_1).$$

Since $m = 2a$ and $(x_1, y_1) = (a, a^2)$ is the point of the line, we get

$$\boxed{y - a^2 = 2a(x - a)}$$

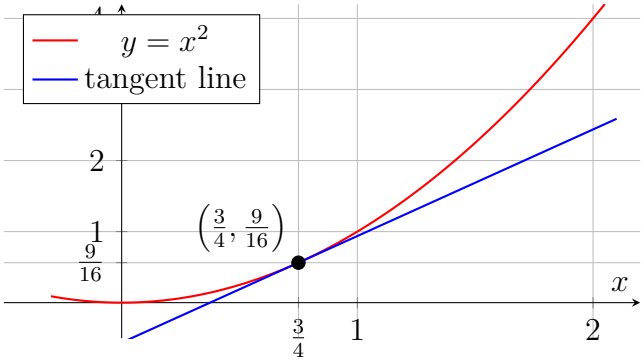
Example

Find the slope and the equation of the tangent line to

$$y = x^2$$

at the point $\left(\frac{3}{4}, \frac{9}{16}\right)$.

Example 1: Graphically



Find the slope

At the point

$$\left(\frac{3}{4}, \frac{9}{16}\right),$$

the x-coordinate is $x = \frac{3}{4}$. Therefore,

$$\boxed{m = 2\left(\frac{3}{4}\right) = \frac{3}{2}}$$

T

he equation of the tangent line is

$$y - a^2 = 2a(x - a)$$

where $a = \frac{3}{4}$. So

$$y - \left(\frac{3}{4}\right)^2 = \frac{3}{2}\left(x - \frac{3}{4}\right)$$

hence

$$\boxed{y - \frac{9}{16} = \frac{3}{2}\left(x - \frac{3}{4}\right)}$$

Example

Write the equation of the tangent line to the graph of $y = x^2$ at the point where $x = -0.7$.

Find the equation

We apply the formula

$$y - a^2 = 2a(x - a)$$

for $a = -0.7$. Hence, the equation of the tangent line of $y = x^2$ at $x = -0.7$ is

$$y - (-0.7)^2 = 2(-0.7)(x - (-0.7))$$

so

$$\boxed{y - 0.49 = -1.4(x + 0.7).}$$

6 Homework Practice – Lecture 2

Exercise 1 (Homework 1, Problem 4)**.** Find an equation of the given line.

horizontal through $(-1, 2)$.

Exercise 2 (Homework 1, Problem 5)**.** Find the equation of a line that is parallel to the line $y = 7x$ and contains the point $(2, -9)$.

Exercise 3 (Homework 1, Problem 6)**.** Find an equation of the following line.

Parallel to $y = 4x + 3$; x-intercept is $(7, 0)$.

Exercise 4 (Homework 1, Problem 7)**.** When the owner of a gas station sets the price of 1 gallon of unleaded gasoline at \$4, he can sell approximately 20,000 gallons per day. When he sets the price at \$7 per gallon, he can

sell approximately 17,000 gallons. Let $G(x)$ denote the number of gallons of unleaded gasoline sold per day when the price is x dollars. Assume that $G(x)$ is a linear function of x. Approximately how many gallons will be sold per day if the price is set at \$4.50 per gallon?

Exercise 5 (Homework 1, Problem 8)**.** Find the slope of the line tangent to the curve $y = x^2$ at the point $(-0.9, 0.81)$ and then find the corresponding equation of the tangent line.

Exercise 6 (Homework 1, Problem 9)**.** Find the slope of the line tangent to the curve $y = x^2$ at the point $\left(\frac{8}{7}, \frac{64}{49}\right)$ and then find the corresponding equation of the tangent line.