Representation Theory

Concise Notes and Solved Exercises

Konstantinos Bizanos

Columbia, Missouri,
June 2, 2026

Contents

Chapter 1

Summary of the General Theory

We note that every time we say group we will mean finite group, and every time we say field we will mean algebraically closed field.

1.1 Representations of Finite Groups

Question

Let G be a group and k a field. What do we call a representation of G over k?

Απάντηση. A representation of G over k is a group homomorphism $\rho\colon G \to \mathrm{GL}_n(k)$. ■

Question

Are there some useful properties that I should remember?

Απάντηση. Of course! It holds that

- $\rho(1) = \mathrm{I}_n$
- $\rho(gh) = \rho(g)\rho(h)$
- $\rho\left(g^m\right) = \rho(g)^m$

■

Trivial Representation

For every group G and field k there exists the **trivial representation** which is given by

$$\rho_{\text{tr}}\colon G \to \mathrm{GL}_1(k) = k^*, \quad \rho_{\text{tr}}(g) = 1$$

Sign Representation

Let $G = \mathfrak{S}_n$. Then the **signed representation** is defined as follows :

$$\rho_{\text{sign}} \colon \mathfrak{S}_n \to \mathrm{GL}_1(k) = k^*, \quad \rho_{\text{sign}}(\sigma) = \text{sign}(\sigma)$$

Question

When is a representation ρ called faithful?

Απάντηση. A representation ρ is called **faithful** if it is 1-1, that is $\ker \rho = \{1\}$. ■

1.2 The group ring kG

We will define a new ring kG (k - algebra) and we will see that the modules over this ring essentially coincide with the representations of G over k

Question

What do we define as the algebra kG?

Απάντηση. Let G be a finite group and k a field. We define as kG the k - vector space

$$kG = \bigoplus_{g \in G} kg$$

that is the free k - module with basis G. Essentially, every $x \in kG$ is written uniquely in the form

$$x = \lambda_1 g_1 + \cdots + \lambda_n g_n$$

where $\lambda_i \in k$ and $G = \{g_1, \ldots, g_n\}$. ■

Question

How can we define operations on kG so that it really obtains a structure of k - algebra?

Απάντηση.

- **Addition** :

$$\sum_{g \in G} \lambda_g g + \sum_{g \in G} \mu_g g = \sum_{g \in G} (\lambda_g + \mu_g) g$$

- **Scalar Multiplication**

$$\lambda \cdot \left(\sum_{g \in G} \lambda_g g \right) = \sum_{g \in G} (\lambda \lambda_g) \, g, \quad \lambda \in k$$

- **Ring Multiplication**
$$\left(\sum_{g\in G}\lambda_g\, g\right)\cdot\left(\sum_{h\in G}\lambda_h\, h\right)=\sum_{g,h\in G}(\lambda_g\mu_h)\, gh$$
essentially the multiplication is done similarly to the law of the distributive property (but it still needs attention!).

■

Question

Is it true that every $\rho\colon G\to \mathrm{GL}_n(k)$ induces a kG module? If yes, how?

Απάντηση. Every matrix $A\in\mathrm{GL}_n(k)$ is a k - linear automorphism defined as follows :

$$\varphi\colon k^n\to k^n \quad v\mapsto A\cdot v$$

Let $V=k^n$. The representation ρ induces an action of G on V which is defined as

$$g\cdot v=\rho(g)(v)\in V$$

Therefore, for an arbitrary $\sum_{g\in G}\lambda_g g\in kG$ we have that

$$\left(\sum_{g\in G}\lambda_g g\right)\cdot v=\sum_{g\in G}\lambda_g\rho(g)(v)$$

by extending the action of G to the whole ring kG. It is left as an exercise to show that V is indeed a kG - module. ■

Question

Is the converse true? That is, does it hold that for every kG module V (which is a k - vector space as well) with $\dim_k V=n<\infty$, it induces a representation $\rho_V: G\to\mathrm{GL}_n(k)$? And if yes, in what way?

Απάντηση. Let V be a kG - module and let $\mathscr{B}=\{v_1,\dots,v_n\}$ be a basis of V. If $g\in G$, observe that a k - linear map is defined

$$\varphi_g\colon V\to V,\quad v\mapsto g\cdot v$$

which is an automorphism, since $\varphi_{g^{-1}}=\varphi_g^{-1}$. Therefore, if $[g]_{\mathscr{B}}$ is the matrix of φ_g with respect to the basis $\mathscr{G}$, we observe that it is defined

$$\rho\colon G\to\mathrm{GL}_n(k),\quad \rho(g)=[g]_{\mathscr{B}}$$

By establishing the notation (V,ρ_V), we mean that ρ_V is the induced representation of the kG - module V. ■

The regular kG - module

We consider an action of G on kG in the following way

$$g \cdot \left(\sum_{h \in G} \lambda_h h \right) = \sum_{h \in G} \lambda_h gh = \sum_{h \in H} \lambda_{g^{-1}h} h$$

By extending linearly, kG enherits the structure of a kG module. The kG with the above action of kG is called the **regular kG - module** . The induced representation is called the **regular representation** of G.

Permutation Module

We consider $G = \mathfrak{S}_n$ and V a k - vector space of dimension n and $\mathscr{B} = \{v_1, \dots, v_n\}$. An action of G on $\mathscr{B}$ is defined as follows:

$$\sigma \cdot v_i := v_{\sigma(i)}$$

and it is extended to all of V.

Example 1.2.1. Let $G = \mathfrak{S}_3$ and V be a k - vector space of dimension 3 and $\mathscr{B} = \{v_1, v_2, v_3\}$. Then, if $\sigma = (12) \in \mathfrak{S}_3$, we notice that

$$\sigma \cdot v_1 = v_2 \quad \sigma \cdot v_2 = v_1 \quad \sigma \cdot v_3 = v_3$$

In the corresponding induced representation it follows that

$$\rho(\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Question

Let V be a kG - module. When is it called irreducible?

Απάντηση. The V is called an **irreducible** kG - module if there is no $W \neq 0$ nonzero subspace such that it is invariant under the action of G (that is $gW \subseteq W$, for every $g \in G$). Equivalently, it has no nontrivial, proper kG - submodule. ■

Question

Let V, W be two kG - modules and $\varphi \colon V \to W$ a k - linear map. When is φ called kG - linear?

Απάντηση. The φ is called kG **- linear** if for every $g \in G$ and $v \in V$ it holds that

$$\varphi(g \cdot v) = g \cdot \varphi(v)$$

Essentially φ is a homomorphism of kG - modules in the usual way from module theory. ■

Question

Let V, W be two kG - modules and $\varphi\colon V \to W$ a kG - linear map. Do we know some basic submodules of V, W that are induced by φ?

Απάντηση. Yes! It holds that $\ker\varphi \leq V$ and $\mathrm{Im}\varphi \leq W$ are kG - submodules of V and W respectively. Also it holds that

$$V = \ker\varphi \oplus U$$

where $U \leq V$ is a kG - submodule where $U \cong \mathrm{Im}\varphi$ (isomorphic kG - modules). ■

Question

Let (V, ρ) be a kG - module and $(U_1, \rho_1), (U_2, \rho_2)$ two kG - submodules such that $V = U_1 \oplus U_2$. What can we say about the corresponding representation of V?

Απάντηση. Let $\mathscr{B}_1, \mathscr{B}_2$ be bases for U_1, U_2 respectively. Then, it holds that $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ is a basis of V. Then, for every $g \in G$ observe that

$$\rho(g) = [g]_{\mathscr{B}} = \begin{pmatrix} [g]_{\mathscr{B}_1} & 0 \\ 0 & [g]_{\mathscr{B}_2} \end{pmatrix} = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

Many times we also denote it by $\rho = \rho_1 \oplus \rho_2$ direct sum of representations. ■

1.3 Maschke's Theorem and Schur's Lemma

Question

What does Maschke's theorem say?

Απάντηση. Let V be a kG - module and $U \leq V$ a kG - submodule. Then, there exists $W \leq V$ a kG - submodule such that

$$V = U \oplus W$$

In other words, the group ring kG is a semi-simple algebra. ■

Question

Can a kG - module V be decomposed into a direct sum of irreducibles?

Απάντηση. Of course! From Maschke's Theorem it follows that every one decomposes into a direct sum of irreducibles

$$V = U_1 \oplus \cdots \oplus U_n$$

Grouping the pairwise isomorphic factors then we have that

$$V = U_1^{\lambda_1} \oplus \cdots \oplus U_k^{\lambda_k}$$

It turns out that $\lambda_i = \dim_k U_i$, consequently we have that

$$V = U_1^{\dim_k U_1} \oplus \cdots \oplus U_k^{\dim_k U_k}$$

■

Application

Applying the above formula for the regular kG - module and passing to degrees we have that

$$|G| = \dim_k U_1^2 + \cdots + \dim_k U_k^2$$

Question

What does Schur's lemma say?

Απάντηση. If V, W are two irreducible kG - modules and $\varphi \colon V \to W$ is a homomorphism of kG - modules, then $\varphi = 0$ or φ is an isomorphism. An important corollary of this lemma is the following. If V is an irreducible kG - module and $\varphi \colon V \to V$ is a homomorphism of kG - modules, then there exists $\lambda \in k$ such that

$$\varphi = \lambda \cdot \mathrm{Id}_V$$

■

1.4 Representations of Abelian Groups

In representation theory the irreducible representations are like the Lego pieces of a more general construction (representation), as the above corollary of Maschke's theorem indicates. We will try to explain techniques which quite often are quite useful for finding the irreducible representations of a group. From now on we will consider that $k = \mathbb{C}$.

Question

We assume that G is abelian. Which are the irreducible representations of G?

Απάντηση. From Schur's lemma (exercise) it follows that a representation (V, ρ) is irreducible if and only if $\dim_k V = 1$, thus the irreducible representations of an abelian group are all one dimensional. ■

Question

Let G be abelian. How can we find its one-dimensional representations?

Απάντηση. We know, from the abelian groups structure theorem that

$$G = C_{n_1} \oplus \cdots \oplus C_{n_k}$$

where $C_{n_i} = \langle a_i \rangle$ is cyclic of order n_i. If we find all the representations $\rho_i \colon C_{n_i} \to \mathbb{C}^*$ Then, the desired representations will be the

$$\rho \colon G \to \mathbb{C}^*, \quad \rho(m_1 a_1 + \cdots + m_k a_k) = \prod_i^k \rho_i(a_i)^{m_i}$$

■

Question

How do I find the irreducible representations of a cyclic group $\mathrm{C} = \langle a \rangle$ of order n?

Απάντηση. Let $\rho \colon \mathrm{C} \to \mathbb{C}^*$ be an irreducible representation. It is enough to determine the value of $\rho(a)$. We have that

$$\rho(a)^n = \rho\left(a^n\right) = \rho(1) = 1$$

Therefore, $\rho(a)$ is an n–th root of unity. Consequently, if ω is a primitive n–th root of unity, we have that all the 1-dimensional representations of C are the

$$\rho_i \colon \mathrm{C} \to \mathbb{C}^*, \quad \rho_i(a) = \omega^i$$

■

Example 1.4.1. a. We will find the 1-dimensional representations of $\mathbb{Z}_2$. If $\rho \colon \mathbb{Z}_2 \to \mathbb{C}^*$ is a representation, we saw that

$$\rho\left(\overline{1}\right) \in \{-1, 1\}$$

Consequently, there are two irreducible representations of $\mathbb{Z}_2$, namely

$$\rho\left(\overline{1}\right) = 1 \quad \text{and} \quad \rho\left(\overline{1}\right) = -1$$

b. We will find the 1-dimensional representations of $\mathbb{Z}_3$. If $\rho\colon \mathbb{Z}_3 \to \mathbb{C}^*$ is a representation, we saw that

$$\rho\left(\overline{1}\right) \in \left\{1, \omega, \omega^2\right\}$$

where ω is a third root of unity. Consequently, there are three irreducible representations of $\mathbb{Z}_3$.

c. We find the irreducible representations of $G = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. From the above examples we have seen that the irreducible representations of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are the

$$\tau_1\left(\overline{1}_2\right) = 1,\ \tau_2\left(\overline{1}_2\right) = -1,\ \rho_1\left(\overline{1}_3\right) = 1,\ \rho_2\left(\overline{1}_3\right) = \omega,\ \rho_3\left(\overline{1}_3\right) = \omega^2$$

Therefore, G has 6 one-dimensional representations give from the following formula :

$$\rho_{i,j}([n]_2 + [m]_3) = \tau_i([n]_2) \cdot \rho_j([m]_3) = (-1)^{i \cdot n} \cdot \omega^{j \cdot m}$$

1.5 Characters

Question

What is the character of a representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$?

Απάντηση. We define as the **character** of ρ the map

$$\chi\colon G \to \mathbb{C}, \quad \chi(g) = \mathrm{Tr}(\rho(g))$$

Obviously, the character of a $\mathbb{C}G$ - module is defined to be the character of the corresponding induced representation. (The character is independent of the choice of basis). ■

Example 1.5.1. Let $G = \mathfrak{S}_3$ and V be a $\mathbb{C}$ - vector space of dimension 3 and $\mathscr{B} = \{v_1, v_2, v_3\}$. Then, if $\sigma = (12) \in \mathfrak{S}_3$ we observe that

$$\sigma \cdot v_1 = v_2 \quad \sigma \cdot v_2 = v_1 \quad \sigma \cdot v_3 = v_3$$

In the corresponding induced representation it follows that

$$\rho(\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, we have that $\chi(\sigma) = 1$.

Question

Does the character have some noteworthy properties?

Απάντηση. Let $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ be a representation.

i. If (V, ρ) is a representation, then $\chi(1) = \dim_{\mathbb{C}} V$.

ii. Let $g, h \in G$. Then we have that $\rho\left(h^{-1}gh\right) = \rho(h)^{-1}\rho(g)\rho(h)$. Since similar matrices have the same traces, then $\chi\left(h^{-1}gh\right) = \chi(g)$. Therefore, characters are constant on conjugacy classes

$$\mathrm{Cl}(g) = \left\{h^{-1}gh \mid h \in G\right\}$$

iii. If V is a $\mathbb{C}G$ - module and $V = U_1 \oplus \cdots \oplus U_n$ a decomposition of it into $\mathbb{C}G$ - modules, then it holds that

$$\chi_V = \chi_{U_1} + \cdots + \chi_{U_n}$$

iv. If $V \cong U$ are isomorphic $\mathbb{C}G$–modules, then $\chi_V = \chi_U$.

v. Let $g \in G$ be an element of order m. Then, $\chi(g)$ is a sum of m–th roots of unity (use that every matrix over $\mathbb{C}$ is triangularizable).

vi. $\chi\left(g^{-1}\right) = \overline{\chi(g)}$.

■

Character of the Regular $\mathbb{C}G$ - Module

We saw that $\mathbb{C}G$ is a $\mathbb{C}G$–module with basis $\mathscr{B} = G = \{g_1 = 1, \ldots, g_n\}$. If $g \neq 1$ in G we observe that $g \cdot g_i \neq g_i$, therefore the corresponding element in position (i, i) of the matrix $[g]_{\mathscr{B}}$ is 0. Therefore, we have that $\chi_{\text{reg}}(g) = 0$. Therefore, we have that

$$\chi_{\text{reg}}(g) = \begin{cases} |G|, & g = 1 \\ 0, & g \neq 1 \end{cases}$$

Character of the Permutation Module

We consider V a $\mathbb{C}$–vector space with basis $\{v_1, \ldots, v_n\}$. We showed that $\mathfrak{S}_n$ defines an action

$$\sigma \cdot v_i = v_{\sigma(i)}$$

If $\sigma \in \mathfrak{S}_n$, we observe that in the corresponding matrix $\rho(g)$, in position (i, i), it has 1 if $\sigma(i) = 1$, while 0 if $\sigma(i) \neq 0$. Taking Tr on $\rho(g)$ we count the units on the diagonal, then units as many as the i for which $\sigma(i) = i$. Therefore, we have that

$$\chi_V(\sigma) = |\mathrm{Fix}(\sigma)|, \quad \mathrm{Fix}(\sigma) = \{i \in \{1, \ldots, n\} \mid \sigma(i) = i\}$$

1.6 Inner Product and Characters

Question

We consider the $\mathbb{C}$–vector space $\mathscr{F}(G,\mathbb{C})$ of the functions from G to $\mathbb{C}$. Can we define an inner product on $\mathscr{F}(G,\mathbb{C})$?

Απάντηση. We can, yes! If $f, g \in \mathscr{F}(G,\mathbb{C})$, then we define

$$\langle f, h\rangle = \frac{1}{|G|}\sum_{g\in G} f(g)\cdot \overline{h(g)}$$

■

Question

What holds in the special case where f, g are characters of G?

Απάντηση. If χ, ψ are characters of G we saw that they stay constant on the corresponding conjugacy classes, therefore if $\{g_1, \dots, g_k\}$ is a set of representatives of the conjugacy classes of G, then we have that

$$\langle \chi, \psi\rangle = \frac{1}{|G|}\sum_{i=1}^{k} |\mathrm{Cl}(g_i)|\chi(g_i)\cdot \overline{\psi(g_i)}$$

■

Question

How are characters connected with the irreducible representations of G?

Απάντηση. If V, W are two distinct irreducible representations of G it holds that

$$\langle \chi_V, \chi_W\rangle = 0 \quad \text{and} \quad \langle \chi_V, \chi_V\rangle = 1$$

The converse holds as well. If for a representation V it holds that $\langle \chi_V, \chi_V\rangle = 1$, then V is an irreducible $\mathbb{C}G$ - module. It turns out that V, W are isomorphic $\mathbb{C}G$– modules if and only if $\chi_V = \chi_W$. ■

Example 1.6.1. Let $G = \mathfrak{S}_3$ and V be the usual permutation module with basis $\{v_1, v_2, v_3\}$. We have seen that the subspace $U = \langle v_1 + v_2 + v_3\rangle$ is a $\mathbb{C}\mathfrak{S}_3$ - submodule and we consider the $\mathbb{C}\mathfrak{S}_3$–module $W = V/U$ with basis $\overline{v_1}, \overline{v_2}$ and

$$\overline{v_3} = -\overline{v_1} - \overline{v_2}$$

Show that W is an irreducible representation. We note that this particular result holds for every $n \geq 1$.

Απάντηση. It suffices to show that $\langle \chi_W, \chi_W \rangle = 1$. First, we observe that :

$$\rho_W(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rho_W(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \rho_W(123) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

Therefore, we have that

$$\chi_W(1) = 2 \quad \chi_W(12) = 0 \quad \chi_W(123) = -1$$

Therefore, it holds that

$$\langle \chi_W, \chi_W \rangle = \\ \frac{1}{6}\left(|\mathrm{Cl}(1)| \cdot \chi_W(1) \cdot \chi_W(1) + |\mathrm{Cl}(12)| \cdot \chi_W(12) \cdot \chi_W(12) + |\mathrm{Cl}(123)| \cdot \chi_W(123) \cdot \chi_W(123)\right) = 1$$

as we desire. ■

Question

How many irreducible representations does a group G have?

Απάντηση. It is proved using character theory that the number of irreducible representations of a group is equal to the number of its conjugacy classes. ■

Example 1.6.2. We will find the character table of $\mathfrak{S}_3$. The $\mathfrak{S}_3$ has three conjugacy classes, with corresponding representatives $1, (12), (123)$, therefore it has 3 irreducible representations.

$\mathfrak{S}_3$	(1)	(12)	(123)
χ_{tr}	1	1	1
χ_{sign}	1	-1	1
χ_W	2	0	-1

where W is the quotient representation $W = \mathbb{C}^3/U$, where $U = \langle e_1 + e_2 + e_3 \rangle$. We will compute various inner products for reasons of practice.

$$\langle \chi_{\text{tr}}, \chi_W \rangle = \frac{1}{6}\left(\chi_{\text{tr}}(1) \cdot \chi_W(1) \cdot 1 + \chi_{\text{tr}}(12) \cdot \chi_W(12) \cdot 3 + \chi_{\text{tr}}(123) \cdot \chi_W(123) \cdot 2\right) = 0$$

and

$$\langle \chi_W, \chi_W \rangle = \frac{1}{6}\left(\chi_W(1) \cdot \chi_W(1) \cdot 1 + \chi_W(12) \cdot \chi_W(12) \cdot 3 + \chi_W(123) \cdot \chi_W(123) \cdot 2\right) = 1.$$

Thus, using the given character table and the inner product formula one can show that $\rho_{\text{tr}}, \rho_{\text{sgn}}$ and ρ_W are the non–isomorphic irreducible representations of $\mathfrak{S}_3$.

Chapter 2

Exercises in Representation Theory

2.1 Conjugacy Classes and Elements of the Symmetric Group

Question

Let $\sigma = (a_1 \cdots a_k) \in \mathfrak{S}_n$ be a cycle of length k, and let $\tau \in \mathfrak{S}_n$. Can we explictly express $\tau\sigma\tau^{-1}$ in terms of σ and τ?

Απάντηση. We have that

$$\tau\sigma\tau^{-1} = (\tau(a_1) \cdots \tau(a_k))$$

Therefore, every conjugate of a cycle of length k is also a cycle of length k. Conversely, if c_1, c_2 are cycles of length k in $\mathfrak{S}_n$, then they are conjugate. Indeed, if

$$c_1 = (a_1 \cdots a_k) \quad \text{and} \quad c_2 = (b_1 \cdots b_k)$$

then, by considering a $\tau \in \mathfrak{S}_n$ such that $\tau(a_i) = b_i$, the above observation gives the desired result. ∎

Question

What holds for the conjugacy class of an arbitrary $\sigma \in \mathfrak{S}_n$?

Απάντηση. We know that every σ can be written as a product of pairwise disjoint cycles. If, in this expression, we also include cycles of length 1, then σ can be written as $\sigma = c_1 \cdots c_\ell$, where c_i is a cycle of length n_i and

$$\sum_{i=1}^{\ell} n_i = n$$

Then, if $\tau \in \mathfrak{S}_n$, we have that

$$\tau\sigma\tau^{-1} = \tau c_1 \cdots c_\ell \tau^{-1} = \prod_{i=1}^{\ell} \tau c_i \tau^{-1} = c_1' \cdots c_\ell'$$

where c_i' is a cycle of length n_i. The converse also holds; that is, any two permutations σ_1, σ_2 which can be written as products of disjoint cycles of the same lengths are conjugate to each other. ■

Example 2.1.1 (Conjugacy Classes of $\mathfrak{S}_3$). Every permutation in
S_3 is written either as a cycle of length 2 or as a cycle of length 3. Therefore, a set of representatives of the conjugacy classes of
S_3 is

$$\mathrm{Cl}(1) = \{1\} \quad \mathrm{Cl}(12) = \{(12), (13), (23)\} \quad \mathrm{Cl}(123) = \{(123), (132)\}$$

Example 2.1.2 (Conjugacy Classes of $\mathfrak{S}_4$). Every permutation in $\mathfrak{S}_4$ can be written as a product of disjoint cycles

- either as a cycle of length 1, that is, as 1,
- either as a cycle of length 2,
- either as a cycle of length 3,
- either as a cycle of length 4,
- or as a product of disjoint cycles of length 2.

Therefore, a set of representatives of the conjugacy classes of $\mathfrak{S}_4$ is

$$1, (12), (123), (1234), (12)(34)$$

with

$$|\mathrm{Cl}(1)| = 1,\ |\mathrm{Cl}(12)| = 6,\ |\mathrm{Cl}(123)| = 8,\ |\mathrm{Cl}(1234)| = 6,\ |\mathrm{Cl}((12)(34))| = 3$$

Abelianization of $\mathfrak{S}_n$

We saw that, in order to compute the 1-dimensional representations of a group, we must reduce to the corresponding 1-dimensional representations of its abelianization, that is, of the group

$$G_{\text{ab}} = G/[G, G]$$

where $[G, G]$ is the corresponding commutator subgroup. What is the abelianization of $\mathfrak{S}_n$?

Απάντηση. We know that if $H \leq G$ is normal and G/H is abelian, then $[G, G] \subseteq H$. In the case of $\mathfrak{S}_n$, we know that for $n \geq 5$ the group A_n is simple; therefore, by the above observation, we have that $[\mathfrak{S}_n, \mathfrak{S}_n] = A_n$, hence

$$(\mathfrak{S}_n)_{\text{ab}} = \mathfrak{S}_n / A_n = \mathbb{Z}_2$$

Similarly, for $n = 2, \ldots, 4$ it is proved that $[\mathfrak{S}_n, \mathfrak{S}_n] = A_n$. Therefore, for every $n \geq 2$ we have that $(\mathfrak{S}_n)_{\text{ab}} = \mathbb{Z}_2$.

■

2.2 Exercises

Exercise 1

We consider a group G, a one-dimensional representation of G with corresponding homomorphism $\sigma\colon G \to \mathbb{C}^*$, and a $\mathbb{C}G$ - module V, with corresponding homomorphism $\rho\colon G \to \mathrm{GL}_{\mathbb{C}}(V)$.

(a) Show that the map $g \mapsto \sigma(g)\rho(g),\ g \in G$ gives the $\mathbb{C}$ - vector space V the structure of a $\mathbb{C}G$ - module V^σ.

(b) Show that the $\mathbb{C}G$ - module V is simple if and only if the $\mathbb{C}G$ - module V^σ is simple.

Απάντηση. (a) It is enough to show that $\sigma(g)\rho(g) \in \mathrm{GL}_{\mathbb{C}}(V)$, which is immediate since $\sigma(g) \neq 0$ and $\rho(g) \in \mathrm{GL}_{\mathbb{C}}(V)$, and that $\sigma\rho$ is a group homomorphism. Let $g, h \in G$. Then,

$$\sigma\rho(gh) = \sigma(gh)\rho(gh) = \sigma(g)\sigma(h)\rho(g) \circ \rho(h) = [\sigma(g)\rho(g)] \circ [\sigma(h)\rho(h)]\,.$$

(b) Let V be a $\mathbb{C}G$ - module and let U be a $\mathbb{C}$ - vector subspace of V. The subspace U is a $\mathbb{C}G$ - submodule of V if and only if $\rho(g)(U) \subseteq U$, for every $g \in G$. Since U is a $\mathbb{C}$ - subspace and $\sigma(g) \neq 0$, this holds if and only if $\sigma(g)\rho(g)(U) \subseteq U$. Therefore, U is a $\mathbb{C}G$ - submodule of V^σ. The converse is shown in the same way, and this gives the desired result.

■

Exercise 2

We consider the group $\mathfrak{S}_6$ of permutations on 6 symbols.

(a) How many pairwise non-isomorphic irreducible representations does the group $\mathfrak{S}_6$ have over the field $\mathbb{C}$?

(b) How many pairwise non-isomorphic one-dimensional representations does the group $\mathfrak{S}_6$ have over the field $\mathbb{C}$?

(c) The group $\mathfrak{S}_6$ acts on the vector space $V = \mathbb{C}^6 = \bigoplus_{i=1}^{6} \mathbb{C}e_i$ through the linear maps which permute the vectors $e_1, \ldots, e_6$. We consider the invariant subspace $U = \mathbb{C}e$, where $e = \sum_{i=1}^{6} e_i$, and the quotient $\mathbb{C}\mathfrak{S}_6$ - module $W = V/U$. Compute the character χ_W and conclude that the $\mathbb{C}\mathfrak{S}_6$ - module W is simple.

(d) Show that there exists at least one irreducible 5-dimensional representation of $\mathfrak{S}_6$ over the field $\mathbb{C}$ which is not isomorphic to the representation W from part (c) above.

Απάντηση. First, we record the conjugacy classes of the group $\mathfrak{S}_6$, together with their sizes:

(**1**)	$\#\mathrm{Cl}(1) = 1$	(**2**)	$\#\mathrm{Cl}(12) = 15$	(**3**)	$\#\mathrm{Cl}(123) = 40$
(**4**)	$\#\mathrm{Cl}(1234) = 90$	(**5**)	$\#\mathrm{Cl}(12345) = 144$	(**6**)	$\#\mathrm{Cl}(123456) = 120$
(**7**)	$\#\mathrm{Cl}(12)(34) = 45$	(**8**)	$\#\mathrm{Cl}(12)(34)(56) = 15$	(**9**)	$\#\mathrm{Cl}(12)(3456) = 90$
(**10**)	$\#\mathrm{Cl}(123)(456) = 40$	(**11**)	$\#\mathrm{Cl}(12)(345) = 120$		

(a) We know that the number of irreducible representations of $\mathfrak{S}_6$ over $\mathbb{C}$ is equal to the number of conjugacy classes of $\mathfrak{S}_6$. Therefore, the number of irreducible representations is 11.

(b) We know that the number of 1-dimensional representations of $\mathfrak{S}_6$ is equal to $|(\mathfrak{S}_6)_{\text{ab}}| = |\mathfrak{S}_6/A_6| = 2$, since $D\mathfrak{S}_6 = A_6$. In fact, these two representations are the trivial representation and the sign representation σ.

(c) Let ρ be the corresponding representation of the $\mathbb{C}\mathfrak{S}_6$ - module W. From the following table, the character table of the representation ρ is:

χ_W	(1) 5	(12) 3	(123) 2
	(1234) 1	(12345) 0	(123456) -1
	$(12)(34)$ 1	$(12)(34)(56)$ -1	$(12)(3456)$ -1
	$(123)(456)$ -1	$(12)(345)$ 0	

We know that the inverse of each of the above representatives of the conjugacy classes is conjugate to the representative itself. Therefore, from the above table it follows that

$$\langle \chi_W, \chi_W \rangle = \sum_{\sigma \in \mathfrak{S}_6} \frac{\chi_W\left(\sigma^{-1}\right)\chi_W(\sigma)}{|\mathfrak{S}_6|} = \sum_{[\sigma]\in\mathscr{C}(\mathfrak{S}_6)} \frac{\gamma_\sigma \chi_W^2(\sigma)}{6!} = 1.$$

Here γ_σ is the size of the corresponding conjugacy class. From the last relation we conclude that W

is a simple $\mathbb{C}\mathfrak{S}_6$ - module.

$\rho(1) = I_5$	$\rho(12) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$\rho(123) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\rho(1234) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$\rho(12345) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	$\rho(123456) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$
$\rho(12)(34) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\rho(12)(34)(56) = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$
$\rho(12)(3456) = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$	$\rho(123)(456) = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$
$\rho(12)(345) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	

(d) Using Exercise 1 and the sign representation σ, we define the $\mathbb{C}\mathfrak{S}_6$ - module W^σ, which is simple by part (c). To show that W and W^σ are not isomorphic, it is enough to show that they have different characters. This holds because $\chi_\rho(12) = 3$, whereas $\chi_{\sigma\rho}(12) = -3$, since $\sigma(12)\rho(12) = -\rho(12)$.

■

Exercise 3

We consider the vector space V with basis $\mathscr{B} = \{v_{i,j} \mid 1 \leq i, j \leq 3\}$ and the action of $\mathfrak{S}_3$ on $\mathscr{B}$ given by

$$\sigma \cdot v_{i,j} = v_{\sigma(i),\sigma(j)}$$

Express χ_V as a linear combination of the irreducible characters of $\mathfrak{S}_3$. Is it true that V contains a $\mathbb{C}\mathfrak{S}_3$ - submodule isomorphic to the regular module of $\mathbb{C}\mathfrak{S}_3$?

Απάντηση. First, we compute the character of V. We have that the action of $\sigma \in \mathfrak{S}_3$ gives a 1 on the diagonal in the column corresponding to (i, j) if and only if $\sigma(i) = i$ and $\sigma(j) = j$. Therefore,

- Clearly, $\chi_V(1) = \dim V = 9$.
- Under the action of (12), the only basis element that remains fixed is $v_{(3,3)}$, hence $\chi_V(12) = 1$.
- Under the action of (123), no basis element remains fixed, hence $\chi_V(123) = 0$.

The character table of $\mathfrak{S}_3$ is

$\mathfrak{S}_3$	(1)	(12)	(123)
χ_{tr}	1	1	1
χ_{sign}	1	-1	1
χ_W	2	0	-1

We have

$$\chi_V = a\chi_{\text{tr}} + b\chi_{\text{sign}} + c\chi_W$$

where

$$a = \langle \chi_V, \chi_{\text{tr}} \rangle = 2 \quad b = \langle \chi_V, \chi_{\text{sign}} \rangle = 1 \quad c = \langle \chi_V, \chi_W \rangle = 3$$

■

Exercise 4

(a) Find the 1-dimensional representations of A_4.

(b) Construct the character table of A_4.

Απάντηση. (a) The group A_4 has the following presentation

$$A_4 = \langle a, b \mid a^3, b^2, (ba)^2 \rangle$$

where $a = (123)$ and $b = (12)(34)$. The abelianization of A_4 is

$$(A_4)_{ab} = \langle a, b \mid a^3, b^2, (ba)^2, aba^{-1}b^{-1} \rangle = \langle a \mid a^3 \rangle = \mathbb{Z}_3$$

Therefore, if ω is a primitive root of unity, A_4 has 3 one-dimensional representations, namely

$$\rho_1(a) = 1 \quad \rho_1(b) = 1$$

the representation

$$\rho_2(a) = \omega \quad \rho_2(b) = 1$$

and

$$\rho_3(a) = \omega^2 \quad \rho_2(b) = 1$$

(b) A set of representatives for the conjugacy classes of A_4 is

$$\{1, b, a, a^{-1}\}$$

where $b = (12)(34)$ and $a = (123)$, with $a^{-1} = (132)$. Since we have found 3 irreducible representations and there are 4 in total, we are still looking for one more irreducible representation ρ_4. Its dimension is given by the relation

$$12 = |A_4| = \sum_{i=1}^{4} \dim \rho_i^2 = 3 + \dim \rho_4^2 \Rightarrow \dim \rho_4 = 3$$

It is left to the reader to show that the desired irreducible representation is $W = V/U$, where V is the permutation module and $U = \langle \sum_i v_i \rangle$, by showing that $\langle \chi_W, \chi_W \rangle = 1$. Therefore, the desired character table is

A_4	(1)	$(12)(34)$	(123)	132
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

■

Exercise 5

Is it true that

(a) for every irreducible character χ different from the trivial character, we have

$$\sum_{g \in G} \chi(g) = 0$$

(b) for every irreducible character different from the sign character, we have

$$\sum_{\sigma \in \mathfrak{S}_n} \text{sign}(\sigma) \cdot \chi(\sigma) = 0$$

Απάντηση. The result follows immediately from the fact that distinct irreducible characters, corresponding to non-isomorphic irreducible modules, are orthogonal. ■

Exercise 6

Show that if there exists $g \in G$ such that

$$\sum_{i=1}^{k} |\chi_i(g)|^2 > \frac{|G|}{2}$$

where $\chi_1, \ldots, \chi_k$ are the irreducible characters of G, then $g \in Z(G)$.

Απάντηση. From the known orthogonality relations, we have

$$\frac{|G|}{|\mathrm{Cl}(g)|} = \sum_{i=1}^{k} |\chi_i(g)|^2 > \frac{|G|}{2}$$

Therefore, $|\mathrm{Cl}(g)| < 2$, so $|\mathrm{Cl}(g)| = 1$. Hence, we obtain the desired result. ■

Exercise 7

Show that a group G is abelian if and only if every irreducible representation of G is 1-dimensional.

Απάντηση. First, assume that G is abelian. Let (V, ρ) be an irreducible representation of G and let $U \leq V$ be a subspace. For every $g \in G$, since G is abelian, the map

$$\rho(g) \colon V \to V,\ v \mapsto g \cdot v$$

is a homomorphism of $\mathbb{C}G$ modules. By Schur's Lemma, we have $\rho(g) = \lambda \mathrm{id}_V$. Therefore,

$$g \cdot U = \rho(g)(U) = \lambda \cdot U \leq U$$

Hence, we have shown that every subspace of V is G - invariant, and since V is irreducible, we conclude that $\dim V = 1$.

Conversely, assume that every irreducible representation of G is irreducible. We know that the number of conjugacy classes ℓ is equal to the number of irreducible representations $\rho_1, \ldots, \rho_\ell$. However, we have

$$|G| = \sum_{i=1}^{\ell} \dim \rho_i^2 = \ell$$

and therefore, from the above relation, we conclude that there are $|G|$ conjugacy classes. Hence, each one must contain exactly one element, and so G is abelian. ■

Exercise 8

Let χ be a character of G, where $\chi(g) = 0$ for every $g \neq 1$. Show that $\chi = m\chi_{\text{reg}}$, where χ_{reg} is the character of the regular representation of G.

Απάντηση. Immediate. ■

Exercise 9

Show that the only one-dimensional representations of $\mathfrak{S}_n$ are the trivial representation and the sign representation.

Απάντηση. We know that the 1-dimensional representations of a group G are in one-to-one and onto correspondence with the 1-dimensional representations of $G_{\text{ab}} = \mathfrak{S}_n/[\mathfrak{S}_n, \mathfrak{S}_n]$. It is proved that

$$[\mathfrak{S}_n, \mathfrak{S}_n] = A_n$$

Therefore, $G_{ab} = \mathbb{Z}_2$, and this gives the desired result. ■

Exercise 10

Let V be a $\mathbb{C}G$ - module and let $W \leq V$ be a $\mathbb{C}G$ - subspace. Show that

$$V \cong W \oplus V/W$$

Απάντηση. We consider the map $\varphi\colon V \to V/W$, which is an epimorphism of $\mathbb{C}G$ modules with $\ker \pi = W$ and $\text{Im}\varphi = V/W$, where V/W is a $\mathbb{C}G$ module. By a known theorem from Linear Algebra, we obtain the desired result. ■

Exercise 11

Let $\rho\colon G \to \text{GL}(V)$ be an irreducible representation.

(a) If $\dim_{\mathbb{C}} V \geq 2$ and $v \in V$, show that $\sum_{g \in G} gv = 0$.

(b) If $g \in \text{C}(G)$, show that $\rho(g) = \zeta \cdot \text{Id}_V$, where ζ is a root of unity.

Απάντηση. (a) Since ρ is irreducible and not isomorphic to the trivial representation, use the fact that

$$\langle \chi_{\text{tr}}, \chi_\rho \rangle = 0$$

(b) Since $g \in \text{C}(G)$, it can be shown that $\rho(g)$ is a homomorphism of $\mathbb{C}G$ modules. Therefore, by Schur's Lemma, we have $\rho(g) = \lambda \cdot \text{Id}_V$ with $\lambda \neq 0$. If $o(g) = m$, then observe that

$$\text{I}_n = \rho(1) = \rho\left(g^m\right) = \left(\rho(g)\right)^m = \lambda^m \cdot \text{I}_n$$

hence λ is an m - th root of unity.

■

Exercise 12

We consider a representation $\rho\colon G \to \text{GL}(V)$ of dimension d with character χ. Show that $\chi(g) = d$, for every $g \in G$, if and only if $\rho(g)$ is the identity automorphism of V, for every $g \in G$.

Απάντηση. If $\rho(g)$ is the identity automorphism of V, for every $g \in G$, then one direction is immediate.

First, we can show that $\chi(g) = d \cdot \zeta_g$, where ζ_g is a root of unity. Indeed, since $\chi(g) = d$, it follows that if $\lambda_{1g}, \dots, \lambda_{dg}$ are the eigenvalues of $\rho(g)$, then

$$\lambda_{ig} = \zeta_g$$

Since $\rho(g)^{o(g)} = \mathrm{I}_d$, we have that $\rho(g)$ is diagonalizable, and in fact $\rho(g) = \zeta_g \cdot \mathrm{I}_d$. Since $\zeta_g = 1$ ($\chi(g) = d = d\zeta_g$), we obtain the desired result. ■

Exercise 13

(a) Classify, up to isomorphism, all semisimple $\mathbb{C}$ - algebras of dimension 10. Can each of them arise as a group algebra $\mathbb{C}G$ of a group G of order 10?

(b) Does there exist a finite group $G \cong \mathbb{Z}_{10}$ such that $\mathbb{C}G \cong \mathbb{C}\mathbb{Z}_{10}$?

(c) From now on, we consider the dihedral group

$$\mathrm{D}_{10} = \left\langle r, s \mid r^5 = s^2 = (sr)^2 = 1 \right\rangle$$

which is the symmetry group of the regular pentagon. The conjugacy classes of D_{10} are:

$$\{1\}\,,\ \left\{s, sr, sr^2, sr^3, sr^4\right\}\,,\ \left\{r, r^4\right\}\,,\ \left\{r^2, r^3\right\}$$

i. Show that the map

$$\rho\colon \mathrm{D}_{10} \to \mathrm{GL}_2(\mathbb{C}), \quad \rho(r) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} \quad \rho(s) = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ \sin\vartheta & -\cos\vartheta \end{pmatrix}$$

where $\vartheta = 2\pi/5$, is a representation of D_{10}.

ii. Show that the above representation is irreducible.

iii. Construct the character table of D_{10}.

iv. Using the character table of D_{10}, find the normal subgroups of D_{10}.

v. Let χ be the character of a representation of D_{10} over $\mathbb{C}$. Show that if $\chi(s) = 0$, then $\chi(1)$ is even.

vi. Is D_{10} solvable?

Απάντηση. (a) If R is a semisimple $\mathbb{C}$ - algebra, then by the Wedderburn-Artin Theorem we have

$$R \cong \mathbb{M}_{n_1}(\mathbb{C}) \times \cdots \times \mathbb{M}_{n_r}(\mathbb{C})$$

Therefore,

$$n_1^2 + \cdots + n_r^2 = 10$$

Hence, we have

$$n_1 = \cdots = n_{10} = 1 \quad \text{or} \quad n_1 = n_2 = 2 \text{ and } n_3 = n_4 = 1 \quad \text{or} \quad n_1 = 3 \text{ and } n_2 = 1$$

Thus, there are three semisimple $\mathbb{C}$ - algebras up to isomorphism. Now, the only groups of order 10, up to isomorphism, are

$$\mathbb{Z}_{10} \quad \text{and} \quad \mathrm{D}_{10}$$

The first case is isomorphic to the algebra $\mathbb{C}\mathbb{Z}_{10}$. Since D_{10} has 4 conjugacy classes, the algebra $\mathbb{C}\mathrm{D}_{10}$ has 4 irreducible representations, two of dimension 1 and two of dimension 2 (this will be shown in the questions below). Therefore, the second case is isomorphic to $\mathbb{C}\mathrm{D}_{10}$.

(b) In order for $\mathbb{C}G \cong \mathbb{C}\mathbb{Z}_{10}$ to hold, every irreducible representation of $\mathbb{C}G$ would have to be 1-dimensional; equivalently, G would have to be abelian. The only abelian group of order 10 is $\mathbb{Z}_{10}$, and therefore there is no G with the desired properties.

(c) i. Immediate.

ii. First, we find the 1-dimensional representations of D_{10}. Thus, we reduce to the 1-dimensional representations of

$$(\mathrm{D}_{10})_{ab} = \left\langle r, s \mid r^5 = s^2 = (sr)^2 = 1,\ sr = rs \right\rangle = \left\langle s \mid s^2 = 1 \right\rangle = \mathbb{Z}_2$$

Hence, we conclude that the 1-dimensional representations of D_{10} are

$$\rho_1(r) = 1 \quad \text{and} \quad \rho_1(s) = 1$$

and

$$\rho_2(r) = 1 \quad \text{and} \quad \rho_2(s) = -1$$

For ρ, we initially have

$$\chi_\rho(1) = 2,\ \chi_\rho(r) = 2\cos\vartheta,\ \chi_\rho(s) = 0,\ \chi_\rho\left(r^2\right) = 2\cos^2\vartheta - 2\sin^2\vartheta$$

If ρ were not irreducible, then it would be of the form

$$\rho = \rho_i \oplus \rho_j$$

with $i, j = 1, 2$. Then we would have

$$\chi_\rho = \chi_{\rho_1} + \chi_{\rho_2}$$

Combining the above, we arrive at a contradiction.

iii. The group D_{10} has 4 conjugacy classes, hence 4 irreducible representations. Since we know 3 of them, we can construct the character table of D_{10} using the orthogonality relations. First, we have

$$10 = |\mathrm{D}_{10}| = \sum_{i=1}^{4} \dim\rho_i^2 = 6 + \dim\rho_4^2 \Rightarrow \dim\rho_4 = 2$$

D_{10}	1	s	r	r^2
χ_1	1	1	1	1
χ_2	1	-1	1	1
$\chi_3 = \chi_\rho$	2	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
χ_4	2	a	b	c

From the relations $\langle \chi_4, \chi_i \rangle = 0$ for $i = 1, 2, 3$, we obtain the following system:

$$\begin{aligned} &a + 2b + 2c = -2 \\ &-a + 2b + 2c = -2 \\ &2b\left(\frac{-1+\sqrt{5}}{2}\right) + 2c\left(\frac{-1-\sqrt{5}}{2}\right) = -4 \end{aligned}$$

From this it follows that

$$a = 0 \quad b = \frac{-1-\sqrt{5}}{2} \quad c = \frac{-1+\sqrt{5}}{2}$$

Hence, the desired character table is

D_{10}	1	s	r	r^2
χ_1	1	1	1	1
χ_2	1	-1	1	1
$\chi_3 = \chi_\rho$	2	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
χ_4	2	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$

iv. If $N_i = \ker \chi_i$, then we have

$$N_1 = \mathrm{D}_{10},\ N_2 = \langle r \rangle,\ N_3 = N_4 = \{1\}$$

Every normal subgroup of D_{10} is an intersection of the above subgroups. Therefore, the only normal subgroups of D_{10} are the above.

v. The character χ can be written as a linear combination of $\chi_1, \ldots, \chi_4$, hence

$$\chi = n_1\chi_1 + n_2\chi_2 + n_3\chi_3 + n_4\chi_4$$

We have

$$\chi(s) = 0 \Leftrightarrow n_1 - n_2 = 0 \Leftrightarrow n_1 = n_2 = n$$

Therefore,

$$\chi(1) = \sum_{i=1}^{4} n_i\chi_i(1) = 2n + 2n_3 + 2n_4$$

and so we obtain the desired result.

vi. Since $|\mathrm{D}_{10}| = 2 \cdot 5$, by Burnside's Theorem it is solvable.

■

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