# All Modules Have Flat Covers

M.Sc. Thesis,

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Athens,

March 16, 2025

# CONTENTS

# ABSTRACT

In this thesis, we demonstrate a detailed proof of Bican, Bachir, and Enochs's results, establishing that each module has a flat cover. Historically, the notion of flat modules was introduced by J.-P. Serre in 1955-1956. A few years later, when injective envelopes had already been studied, the dual notion of injective envelopes, known as projective covers, was investigated. H. Bass, in his 1959 thesis, introduced the concept of projective covers and described the appropriate rings in which each module has a projective cover (left/right perfect rings). Following this result, the question arose of when a module has a **flat cover**.

After many years, significant progress was made by J. Xu, who revived interest in this open problem. The problem was finally solved and presented in Enochs's paper, "All Modules Have Flat Covers", in which the use of certain lemmas proved by Eklof—stated in his paper, "How to Make Ext Vanish" played a crucial role.

- In the first chapter, we introduce the notion of extension of modules $E(A, B)$, we prove some useful lemmas and we present the relation between extension $E(A, B)$ and the abelian group $\mathrm{Ext}^1_R(A, B)$.

- In the second chapter, we provide a summary of envelopes (covers), special envelopes (special covers), and their connections to the concept of cotorsion theories.

- In Chapter 3, we introduce the concepts of flat modules, pure submodules, and their relationship. We conclude the chapter with several characterizations of pure exact sequences. Additionally, we provide an extended review of ordinal numbers, focusing on transfinite induction, which is heavily utilized in the remaining material of this chapter and in Chapter 5.

- In Chapter 4, we present a detailed discussion of Eklof's results, which include

essential techniques for vanishing the Ext functor. Furthermore, using Eklof's Lemma, we prove a core theorem of this thesis: that every cotorsion theory cogenerated by a set of modules is complete. This result is crucial, as we demonstrate that the flat cotorsion theory is complete. Consequently, the existence of a flat precover guarantees the existence of a flat cover for any arbitrary module.

- Finally, in Chapter 5, we present Xu's result, which is pivotal to proving the desired outcome. We define the concept of flat cotorsion theory and establish the central theorem of the thesis.

**Acknowledgment.** I would like to thank my supervisor, Ioannis Emmanouil, for his invaluable support, guidance, and patience with my numerous questions during the preparation of this thesis.

Athens,
March 16, 2025

# CHAPTER 1

# EXTENSIONS OF MODULES

## 1.1 Extensions

Let $A$ and $B$ be two $R$-modules. We aim to consider all $R$-modules $E$ such that $B$ is a submodule of $E$ and $E/B \cong A$.

**Definition 1.1.** A short exact sequence

$$0 \to B \xrightarrow{\kappa} E \xrightarrow{\nu} A \to 0 \tag{1.1}$$

is called an **extension** of $A$ by $B$.

**Definition 1.2.** Two extensions $0 \to B \to E_1 \to A \to 0$ and $0 \to B \to E_2 \to A \to 0$ are called **equivalent** if there exists a homomorphism $\xi : E_1 \to E_2$ such that the following diagram is commutative:

$$\begin{array}{ccccc} B & \rightarrowtail & E_1 & \twoheadrightarrow & A \\ \| & & \downarrow{\scriptstyle \xi} & & \| \\ B & \rightarrowtail & E_2 & \twoheadrightarrow & A \end{array}$$

**Observation 1.1.** From the above diagram, it is evident that $\xi$ is an isomorphism, so it is straightforward to show that equivalence of extensions defines an equivalence relation on the set of extensions of $A$ by $B$. We denote this set by $E(A, B)$.

**Observation 1.2.** The set $E(A, B)$ contains at least one element, the extension

$$0 \to B \xrightarrow{i_B} A \oplus B \xrightarrow{\pi_A} A \to 0.$$

Any extension of $A$ by $B$ equivalent to the above extension is called a **split extension** or **trivial extension** of $A$ by $B$. We denote the trivial extension by 0.

**Lemma 1.1.** An $R$-module $A$ is projective if and only if $E(A, B) = \{0\}$ for every $R$-module $B$.

*Proof.* Assume that $A$ is projective. Let $B$ be an arbitrary module and let

$$0 \to B \xrightarrow{\kappa} E \xrightarrow{\nu} A \to 0$$

be an extension of $A$ by $B$. Since $A$ is projective, there exists a homomorphism $\sigma \in \mathrm{Hom}_R(A, E)$ such that the following diagram commutes:

$$\begin{array}{ccccc} E & \xrightarrow{\nu} & A & \longrightarrow & 0 \\ {\scriptstyle\sigma}\uparrow & \nearrow_{\mathrm{id}} & & & \\ A & & & & \end{array}$$

Therefore, the above extension splits. Conversely, assume that $E(A, B) = \{0\}$ for every module $B$. Let $\mu : M \to N$ be an epimorphism and $\varphi : A \to N$. We consider the pullback

$$P = \{(x, y) \in M \oplus A \mid \mu(x) = \varphi(y)\}$$

of the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\mu} & N & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle\varphi} & & \\ & & A & & \end{array}$$

This gives the short exact sequence

$$0 \to \ker \mu \to P \xrightarrow{\pi_A} A \to 0,$$

which splits. Thus, there exists a homomorphism $\sigma : A \to P$ such that $\pi_A \circ \sigma = \mathrm{id}_A$. If $\psi$ is the composition

$$A \xrightarrow{\sigma} P \xrightarrow{\pi_M} M,$$

then $\varphi = \mu \circ \psi$. □

**Lemma 1.2.** The square

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow{\psi} & X \end{array} \tag{1.2}$$

is a pullback diagram if and only if the following sequence is exact:

$$0 \to Y \xrightarrow{\{\alpha,\beta\}} A \oplus B \xrightarrow{\langle\varphi,-\psi\rangle} X,$$

where

$$\{\alpha, \beta\}(y) = (\alpha(y), \beta(y)) \quad \text{and} \quad \langle\varphi, -\psi\rangle(a, b) = \varphi(a) - \psi(b).$$

*Proof.* The universal property of the pullback diagram of $(\varphi, \psi)$ is shown to be equivalent to the universal property of the kernel of $\langle\varphi, -\psi\rangle$. □

**Lemma 1.3.** If diagram 1.2 is a pullback diagram, then

(a) $\ker\alpha \xrightarrow{\cong} \ker\psi$ via $\beta$,

(b) if $\psi$ is an epimorphism, then $\alpha$ is also an epimorphism.

*Proof.* (a)
- Let $y \in \ker\alpha$. By the commutativity of the diagram, we have $\psi\beta(y)$, i.e., $\beta(y) \in \ker\psi$, so $\beta|_{\ker\alpha} : \ker\alpha \to \ker\psi$ is well-defined.
- Since $\{\alpha, \beta\}$ is injective, it follows that $\beta|_{\ker\alpha}$ is a monomorphism.
- For surjectivity, if $b \in \ker\psi$, then $\langle\varphi, -\psi\rangle(0, b) = 0$. Therefore, there exists $y \in Y$ such that $\beta(y) = b$ and $\alpha(y) = 0$, which proves the desired result.

(b) Let $a \in A$. Then there exists $b \in B$ such that $\psi(b) = \varphi(a)$, since $(a, b) \in \ker\langle\varphi, -\psi\rangle = \operatorname{Im}\{\alpha, \beta\}$. The desired result follows immediately from the previous observation.

□

**Lemma 1.4.** Consider the diagram

$$\begin{array}{ccccc} B & \overset{\kappa'}{\rightarrowtail} & E' & \overset{\nu'}{\twoheadrightarrow} & A' \\ \| & & \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\alpha} \\ B & \overset{\kappa}{\rightarrowtail} & E & \overset{\nu}{\twoheadrightarrow} & A \end{array}$$

If the rows are exact and the diagram commutes, then the right-hand square is a pullback diagram.

*Proof.* • Consider the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{\varepsilon} & A' \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\alpha} \\ E & \xrightarrow{\nu} & A \end{array}$$

By Lemma 1.3, we know that $\varepsilon$ is an epimorphism and $\ker \varepsilon \cong B$.

- Therefore, we obtain the extension $0 \to B \to P \to A' \to 0$. Showing that the extensions $0 \to B \xrightarrow{\kappa} E' \xrightarrow{\nu} A'$ and $0 \to B \to P \to A' \to 0$ are equivalent implies $Y \cong P$, and the desired result follows.

□

The reader is encouraged to prove the dual results of the above lemmas.

## 1.2 $E(A, B)$ and $\text{Ext}^1(A, B)$

Let $A$ be an arbitrary $R$ - module. We can take $P$ a projective module and a short exact sequence

$$0 \to K \xrightarrow{i} P \xrightarrow{\pi} A \to 0$$

For every $R$ - module $B$ we can apply then functor $\text{Hom}_R(-, B)$ to the above sequence

$$0 \to \text{Hom}_R(A, B) \xrightarrow{\pi^*} \text{Hom}_R(P, B) \xrightarrow{i^*} \text{Hom}_R(K, B)$$

We define

$${}_\pi\text{Ext}^1(A, B) := \text{coker}\left(\text{Hom}_R(P, B) \xrightarrow{i^*} \text{Hom}_R(K, B)\right)$$

**Proposition 1.1.** The isomophism class of ${}_\pi\text{Ext}^1(A, B)$ is independent from the choice of $\pi$, for every $R$ - module $B$. Therefore for simplicity we write $\text{Ext}^1(A, B)$ for ${}_\pi\text{Ext}^1(A, B)$.

**Theorem 1.1.** There is a bijection $E(A, B) \cong \text{Ext}^1(A, B)$ preserving 0.

*Proof.* Let $\varphi \colon K \to B$. We shall show that this morphism corresponds to an extension

$$0 \to B \to Q_\varphi \to A \to 0$$

With the above denotation consider the pushout

$$Q_\varphi = P \oplus B / \{(i(k), -\varphi(k)) \mid k \in K\}$$

of diagram

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & B \\ \downarrow{\scriptstyle i} & & \\ P & & \end{array}$$

If we define $\psi\colon Q_\varphi \to A$, $\psi([p,b]) = \pi(p)$, can be easily seen that the following sequence

$$0 \to B \xrightarrow{\pi_B} Q_\varphi \xrightarrow{\psi} A \to 0$$

if an extension of $A$ by $B$. Is left to the reader to show that if $\varphi, \varphi' \in \mathrm{Im} i^*$, then the induced extensions

$$0 \to B \xrightarrow{\pi_B} Q_\varphi \xrightarrow{\psi} A \to 0 \quad \text{and} \quad 0 \to B \xrightarrow{\pi_B} Q_{\varphi'} \xrightarrow{\psi'} A \to 0$$

are equivalent. Conversely, let

$$0 \to B \xrightarrow{\kappa} E \xrightarrow{\nu} A \to 0$$

be an extension of $A$ by $B$. By projectivity of $P$ there are $\varphi\colon P \to E$ and $\psi\colon K \to B$ s.t. the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & P & \xrightarrow{\pi} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & B & \xrightarrow{\kappa} & E & \xrightarrow{\nu} & A & \longrightarrow & 0 \end{array}$$

Is left to the reader to show that if two extensions are equivalent, then for the induced maps $\psi, \psi'\colon K \to B$ is true that $\psi - \psi' \in \mathrm{Im} i^*$.

Finally, it can be easily seen that if $\varphi\colon K \to B$ s.t. there if $\Phi\colon P \to B$ and $\Phi|_K = \Phi \circ i = \varphi$, then the induced extension is the trivial extension. This shows that the above bijection preserves 0. □

**Corollary 1.1.** The set $E(A,B)$ has the structure of an abelian group

*Proof.* See [1] section 7.2.1 for an explicit description of the group structure in terms of extensions. □

# CHAPTER 2

# APPROXIMATIONS OF MODULES

## 2.1 Preenvelopes and Envelopes

**Definition 2.1.** Let $M$ be an $R$-module, and let $\mathscr{C}$ be a class of $R$-modules that is closed under isomorphic images and direct sums. A map $f \in \mathrm{Hom}(M, C)$, where $C \in \mathscr{C}$, is called a $\mathscr{C}$**-preenvelope** if, for every $f' \in \mathrm{Hom}(M, C')$ with $C' \in \mathscr{C}$, there exists $g \in \mathrm{Hom}(C, C')$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & C \\ & {\scriptstyle f'}\searrow & \downarrow{\scriptstyle g} \\ & & C' \end{array}$$

The $\mathscr{C}$-preenvelope $f$ is called a $\mathscr{C}$**-envelope** of $M$ if $g$ is an automorphism whenever $g \in \mathrm{Hom}_R(C, C)$ and $f = gf$.

**Example 2.1.** Let $\mathscr{I}_0$ be the class of injective $R$-modules. If $M$ is an $R$-module and $E(M)$ is the injective hull of $M$, then the inclusion map $i : M \hookrightarrow E(M)$ is a $\mathscr{C}$**-envelope**.

*Proof.* Let $f' \colon M \to I$ be an $R$-homomorphism, where $I$ is an injective $R$-module. Since $I$ is injective and $i$ is a monomorphism, there exists $g \in \mathrm{Hom}_R(E(M), I)$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \overset{i}{\hookrightarrow} & E(M) \\ & \underset{f'}{\searrow} & \downarrow g \\ & & I \end{array}$$

Now let $g\colon E(M) \to E(M)$ be an $R$-homomorphism such that $i = gi$. Since $E(M)$ is an essential extension of $M$, it follows that $g$ is a monomorphism. Thus, there exists $g' \in \mathrm{End}_R(E(M))$ such that:

$$\begin{array}{ccc} E(M) & \overset{g}{\hookrightarrow} & E(M) \\ & \underset{\mathrm{id}}{\searrow} & \downarrow g' \\ & & E(M) \end{array}$$

It is evident that $g'i = i$, and similarly, we conclude that $g' \in \mathrm{Aut}_R(E(M))$. Hence, this implies that $g\colon E(M) \to E(M)$ is an $R$-automorphism. □

**Observation 2.1.** In general, given an $R$-module $M$, there may be several different $\mathscr{C}$-preenvelopes for $M$ but no $\mathscr{C}$-envelopes. However, the next lemma shows that if a $\mathscr{C}$-envelope exists, it is the minimal $\mathscr{C}$-preenvelope in the following sense:

**Lemma 2.1.** Let $f : M \to C$ be a $\mathscr{C}$-envelope, and let $f' : M \to C'$ be a $\mathscr{C}$-preenvelope. Then:

(a) $C' = D \oplus D'$, $\mathrm{Im} f' \subseteq D$, and the map $f' : M \to D$ is a $\mathscr{C}$-envelope of $M$.

(b) The map $f'$ is a $\mathscr{C}$-envelope if and only if it has no proper direct summand contained in $\mathrm{Im} f'$.

*Proof.* (a) Since $f$ and $f'$ are $\mathscr{C}$-preenvelopes, there exist maps $g : C \to C'$ and $g' : C' \to C$ such that $gf = f'$ and $g'f' = f$. Therefore, the following commutative diagram arises:

$$\begin{array}{ccc} M & \overset{f}{\longrightarrow} & C \\ & \underset{f}{\searrow} & \downarrow g'g \\ & & C \end{array}$$

Since $f$ is a $\mathscr{C}$-envelope, $g'g$ is an automorphism. Thus, $g$ is a monomorphism, and $g'$ is an epimorphism. Let $D = \mathrm{Im} g \cong C$ and observe that $\mathrm{Im} f' \subseteq D$. If

$$D' = C'/\mathrm{Im} g$$

then the short exact sequence

$$0 \to D \hookrightarrow C' \to D' \to 0$$

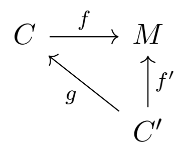
splits, since $g'g$ is an automorphism, implying that $C' \cong D \oplus D'$.

(b) The desired result follows directly from (a).

□

## 2.2 Precovers and Covers

**Definition 2.2.** Let $\mathscr{C} \subseteq \mathsf{Mod}\text{-}R$ be a class of modules closed under isomorphic images and direct summands. Let $M \in \mathsf{Mod}\text{-}R$. A map $f \in \mathrm{Hom}_R(C, M)$, where $C \in \mathscr{C}$, is called a $\mathscr{C}$**-precover** of $M$ if, for each $C' \in \mathscr{C}$ and for each $f' \in \mathrm{Hom}_R(C', M)$, there exists $g \in \mathrm{Hom}_R(C', C)$ such that the following diagram commutes:




A $\mathscr{C}$-precover $f \in \mathrm{Hom}_R(C, M)$ is called a $\mathscr{C}$**-cover** of $M$ if $fg = f$ and $g \in \mathrm{End}_R(C)$ implies that $g \in \mathrm{Aut}_R(C)$.

**Observation 2.2.** By the preceding definition, it is evident that $f \in \mathrm{Hom}_R(C, M)$ is a $\mathscr{C}$-precover if and only if $f$ induces a surjective abelian group homomorphism:

$$\mathrm{Hom}_R(C', C) \xrightarrow{f_*} \mathrm{Hom}_R(C', M).$$

**Example 2.2.** Let $\mathscr{P}_0$ be the category of projective $R$-modules. Then every $M \in \mathrm{Mod} - R$ has a $\mathscr{P}_0$-precover. Many questions arise here: Is it true that every module has a $\mathscr{P}_0$-cover? *Moreover, if we generalize this, is it true that every module has a flat cover?* To answer this question, we need to take a different approach.

**Lemma 2.2.** Let $f\colon C \to M$ be a $\mathscr{C}$-cover of $M$, and let $f'\colon C' \to M$ be any $\mathscr{C}$-precover of $M$. Then:

(a) $C' = D \oplus D'$, where $D \subseteq \ker f'$, and the restriction $f'|_{D'}$ is a $\mathscr{C}$-cover of $M$.

(b) $f'$ is a $\mathscr{C}$-cover of $M$ if and only if $C'$ has no nonzero direct summands contained in $\ker f'$.

*Proof.* This follows dually from the proof of Lemma 2.1. □

## 2.3 Cotorsion Theories

**Definition 2.3.** Let $\mathscr{C} \subseteq \text{Mod} - R$. Define

$$\mathscr{C}^{\perp} = \left\{ N \in \text{Mod} - R \mid \text{Ext}_R^1(C, N) = 0 \text{ for all } C \in \mathscr{C} \right\}$$

and

$$^{\perp}\mathscr{C} = \left\{ N \in \text{Mod} - R \mid \text{Ext}_R^1(N, C) = 0 \text{ for all } C \in \mathscr{C} \right\}.$$

**Definition 2.4.** Let $M \in \text{Mod} - R$. A $\mathscr{C}$-preenvelope $f\colon M \to C$ is called **special** if the following sequence is exact:

$$0 \to M \xrightarrow{f} C \to \text{coker} f \to 0,$$

and coker$f \in^{\perp} \mathscr{C}$. Dually, a $\mathscr{C}$-precover $f\colon C \to M$ is called **special** if the following sequence is exact:

$$0 \to \ker f \hookrightarrow C \xrightarrow{f} M \to 0,$$

and $\ker f \in \mathscr{C}^{\perp}$.

**Lemma 2.3.** Let $M \in \text{Mod-}R$ and let $\mathscr{C} \subseteq \text{Mod-}R$ be closed under extensions.

(a) If $\mathscr{I}_0 \subseteq \mathscr{C}$ and $f\colon M \to C$ is a $\mathscr{C}$-envelope, then $f$ is special.

(b) If $\mathscr{P}_0 \subseteq \mathscr{C}$ and $f\colon C \to M$ is a $\mathscr{C}$-cover, then $f$ is special.

*Proof.* (a) We want to show that $f$ is injective and that coker$f \in^{\perp} \mathscr{C}$.

- To prove injectivity, consider the injective hull $M \xrightarrow{i} E(M)$. Then there exists $k\colon C \to E(M)$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & C \\ & \searrow^{i} & \downarrow k \\ & & E(M) \end{array}$$

  Since $i$ is injective and $i = k \circ f$, it follows that $f$ is injective.
- Now we show that $D = \text{coker} f \in^{\perp} \mathscr{C}$. Let $C' \in \mathscr{C}$. Showing that

$$\text{Ext}_R^1\left(D, C'\right) = 0$$

  is equivalent to proving that every short exact sequence

$$0 \to C' \to X \xrightarrow{h} D \to 0$$

splits. We consider the following pullback diagram:

$$\begin{array}{ccc} P & \xrightarrow{\beta} & X \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{g} & D \end{array}$$

From Lemma 1.3, we have the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & C' & = & C' & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & X & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\delta}\uparrow\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & M & \xrightarrow{f} & C & \xrightarrow{g} & D & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

- Since $\mathscr{C}$ is closed under extensions and $C, C' \in \mathscr{C}$, then $P \in \mathscr{C}$.
- Since $f$ is a $\mathscr{C}$-envelope, there exists $\delta\colon C \to P$ such that $\alpha = \delta f$. Then we have
$$f = \gamma\alpha = \gamma\delta f$$
implying that $\gamma\delta$ is an automorphism of $C$.
- Define $i\colon D \to X$ by
$$i(g(c)) = \beta\delta(\gamma\delta)^{-1}(c)$$
The map $i$ is well-defined via the above commutative diagram. It is straightforward that $hi = \mathrm{id}_D$, and thus the sequence splits.

□

**Definition 2.5.** A pair $(\mathscr{A}, \mathscr{B})$ of module classes is called a **cotorsion theory** if:

$$\mathscr{A} = {}^{\perp}\mathscr{B} \quad \text{and} \quad \mathscr{B} = \mathscr{A}^{\perp}.$$

If $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ is a cotorsion theory, then the class $\mathscr{K}_{\mathfrak{C}} = \mathscr{A} \cap \mathscr{B}$ is called the **kernel** of $\mathfrak{C}$.

**Example 2.3.** Let $\mathscr{C}$ be any class of modules. Then the pair

$$\mathfrak{S}_{\mathscr{C}} = \left( {}^{\perp}\mathscr{C}, \left( {}^{\perp}\mathscr{C} \right)^{\perp} \right)$$

is a cotorsion theory, called the cotorsion theory **generated** by the class $\mathscr{C}$.

*Proof.* For any class of modules $\mathscr{A}$, it is evident that $\mathscr{A} \subseteq^{\perp} (\mathscr{A}^{\perp})$; therefore, ${}^{\perp}\mathscr{C} \subseteq^{\perp} \left( \left( {}^{\perp}\mathscr{C} \right)^{\perp} \right)$.

For the converse relation, suppose $M \in^{\perp} \left( \left( {}^{\perp}\mathscr{C} \right)^{\perp} \right)$. This implies that

$$\mathrm{Ext}^1_R(M, X) = 0 \quad \text{whenever} \quad \mathrm{Ext}^1_R(B, X) = 0 \quad \forall B \in^{\perp} \mathscr{C}.$$

Since $\mathrm{Ext}^1_R(M, X) = 0$ for every $X \in \mathscr{C}$, it follows that $M \in^{\perp} \mathscr{C}$. □

**Example 2.4.** Let $\mathscr{C}$ be any class of modules. Then the pair

$$\mathfrak{C}_{\mathscr{C}} = \left( {}^{\perp} \left( \mathscr{C}^{\perp} \right), \mathscr{C}^{\perp} \right)$$

is, similarly to the above example, a cotorsion theory, called the **cotorsion theory cogenerated** by the class $\mathscr{C}$.

**Example 2.5.** By Lemma 1.1, it is easy to see that

$$\mathfrak{S}_{\text{Mod}-R} = (\text{Mod} - R, \mathscr{I}_0) \quad \text{and} \quad \mathfrak{C}_{\text{Mod}-R} = (\mathscr{P}_0, \text{Mod} - R),$$

where $\mathscr{I}_0$ and $\mathscr{P}_0$ are the classes of injective and projective $R$-modules, respectively. These cotorsion theories are called **trivial cotorsion theories** .

The main reason for introducing and studying $\mathscr{C}$-special preenvelopes and $\mathscr{C}$-special precovers is their close relation to cotorsion theories.

**Definition 2.6.** A cotorsion theory $(\mathscr{A}, \mathscr{B})$ is said to have **enough injectives** (resp. **enough projectives** ) if every module $M$ has a special $\mathscr{B}$-preenvelope (or a special $\mathscr{A}$-precover, respectively).

Although these two concepts appear to be different, we will prove that if a cotorsion theory satisfies one, then it satisfies the other and vice versa. In this case, the cotorsion theory is called **complete**.

**Lemma 2.4.** Let $R$ be a ring, and let $\mathfrak{C} = (\mathscr{A}, \mathscr{B})$ be a cotorsion theory of modules. Then the following statements are equivalent:

(a) Every module has a special $\mathscr{A}$-precover.

(b) Every module has a special $\mathscr{B}$-preenvelope.

*Proof.* Assume that every module has a special $\mathscr{A}$-precover. Let $M \in$ Mod-$R$. We will establish the existence of a $\mathscr{B}$-preenvelope of $M$.

There exists a short exact sequence

$$0 \to M \to I \xrightarrow{\pi} F \to 0,$$

where $I$ is an injective module. Thus, there is a special $\mathscr{A}$-precover

$$0 \to B \to A \xrightarrow{\rho} F \to 0, \quad A \in \mathscr{A},\ B \in \mathscr{B}.$$

Consider the pullback diagram of $\pi$ and $\rho$:

$$\begin{array}{ccc} P & \longrightarrow & I \\ \big\downarrow{\scriptstyle\gamma} & & \big\downarrow{\scriptstyle\pi} \\ A & \xrightarrow{\rho} & F \end{array}$$

From Lemma 1.3, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \big\downarrow & & \big\downarrow & & \\ & & & & M & = & M & & \\ & & & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & P & \longrightarrow & I & \longrightarrow & 0 \\ & & \| & & {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\pi} & & \\ 0 & \longrightarrow & B & \longrightarrow & A & \xrightarrow{\rho} & F & \longrightarrow & 0 \\ & & & & \big\downarrow & & \big\downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since $B, I \in \mathscr{B}$ and $\mathscr{B}$ is closed under extensions (by an easy application of the long exact sequence), it follows that $P \in \mathscr{B}$. Thus, the short exact sequence

$$0 \to M \to P \xrightarrow{\gamma} A \to 0$$

is a special $\mathscr{B}$-preenvelope. $\square$

# CHAPTER 3

# FLAT MODULES AND PURITY

## 3.1 Flat Modules

**Definition 3.1.** A right module $P_R$ is called **flat** if the functor $P \otimes_R -$ is exact on $R$-Mod. Equivalently, whenever $A \to B$ is injective in $R$-Mod, the induced map

$$P \otimes_R A \to P \otimes_R B$$

is also injective in the category of abelian groups.

**Proposition 3.1.** Let $\varphi\colon R \to S$ be a ring homomorphism, whereby $S$ can be viewed as a left $R$-module. If $P_R$ is $R$-flat, then the right $S$-module

$$P' := P \otimes_R S$$

is $S$-flat.

*Proof.* Let $A \xrightarrow{i} B$ be an injective homomorphism in $S$-Mod. We want to show that

$$P' \otimes_S A \xrightarrow{-\otimes i} P' \otimes_S B$$

is injective.

Since $A, B$ can be viewed as left $R$-modules via $\varphi$, it follows that $P \otimes_R A \to P \otimes_R B$ is injective. We can identify

$$S \otimes_S A \cong A \quad \text{and} \quad S \otimes_S B \cong B$$

as left $R$-modules. Combining these observations implies the desired result. □

**Proposition 3.2.** If

$$P = \bigoplus_{i \in I} P_i \in \mathsf{Mod}\text{-}R$$

then $P$ is flat if and only if each $P_i$ is flat.

*Proof.* The key observation to prove this proposition is that $- \otimes_R A$ is the left adjoint functor of $\mathrm{Hom}_R(A, -)$. Thus, $- \otimes_R A$ preserves colimits. Therefore, there exists an abelian group isomorphism

$$\left( \bigoplus_{i \in I} P_i \right) \otimes_R A \cong \bigoplus_{i \in I} (P_i \otimes_R A) \, .$$

Hence, the desired result follows immediately. □

**Corollary 3.1.** Any projective (right) $R$-module is flat.

*Proof.* Let $R$ be a ring. It is easy to see that the free (right) $R$-module $R$ is flat. Therefore, every free module is flat, as it is a direct sum of copies of $R$. Since every projective module is a direct summand of a free module, the result follows from the preceding proposition. □

**Proposition 3.3.** Let $\{P_i \mid i \in I\}$ be a direct system of right modules over any ring $R$, where $I$ is a directed set. If each $P_i$ ($i \in I$) is flat, then the direct limit module

$$P := \varinjlim P_i$$

is also flat.

*Proof.* Let $A \xrightarrow{j} B$ be an injection in $R$-Mod. For each $i \in I$, the map

$$P_i \otimes_R A \to P_i \otimes_R B$$

is injective. By the construction of direct limits, it follows easily that

$$\varinjlim (P_i \otimes_R A) \longrightarrow \varinjlim (P_i \otimes_R B)$$

is injective. Since $- \otimes_R A$ and $- \otimes_R B$ preserve direct limits, it follows that

$$P \otimes_R A \longrightarrow P \otimes_R B$$

is injective, as desired. □

**Observation 3.1.** Every module is the direct limit of its finitely generated submodules.

*Proof.* Let $M$ be an $R$-module, and let $\mathscr{A} = \{N \leq M \mid N \text{ is finitely generated}\}$. Then $(\mathscr{A}, \subseteq)$ is a direct system, whose direct limit is given by

$$\varinjlim N = \bigcup \mathscr{A} = M.$$

□

**Corollary 3.2.** Let $P \in \text{Mod-}R$ be a module whose every finitely generated submodule is flat. Then $P$ is flat.

*Proof.* The result follows immediately from Proposition 3.3 and Observation 3.1. □

## 3.2 Pure Exact Sequences

**Definition 3.2.** A (short) exact sequence

$$\mathscr{E} : 0 \to A \xrightarrow{\varphi} B \to C \to 0$$

in Mod-$R$ is said to be **pure exact** if $\mathscr{E} \otimes_R C'$ is exact for every $C' \in R\text{-Mod}$. In this case, we say that $\varphi(A)$ is a **pure submodule** of $B$.

**Example 3.1.** Let $\mathscr{E} : 0 \to A \xrightarrow{\varphi} B \to C \to 0$ be a split short exact sequence. Then $\mathscr{E}$ is pure.

*Proof.* Let $C' \in R\text{-Mod}$. Since $\mathscr{E}$ splits, there exists $\psi \in \text{Hom}_R(B, A)$ such that

$$\psi \circ \varphi = \text{id}_A$$

$$0 \to A \underset{\psi}{\overset{\varphi}{\rightleftarrows}} B \longrightarrow C \longrightarrow 0$$

Applying the functor $- \otimes_R C'$ to the above sequence, the injectivity of $\varphi \otimes_R C'$ follows immediately. □

**Example 3.2.** The direct sum of pure exact sequences is pure exact.

*Proof.* Let $B_i \leq A_i$ be pure submodules for each $i \in I$. We shall show that

$$B := \bigoplus_{i\in I} B_i$$

is a pure submodule of

$$A := \bigoplus_{i\in I} A_i$$

Equivalently, for any $C \in R$-Mod, if $B \xrightarrow{j} A$ is the inclusion map, then

$$B \otimes_R C \xrightarrow{j\otimes_R C} A \otimes_R C$$

is injective. Since each square in the following diagram commutes:

$$\begin{array}{ccc} B_i \otimes_R C & \xrightarrow{j_i \otimes_R C} & A_i \otimes_R C \\ \downarrow & & \downarrow \\ \bigoplus_{i\in I}(B_i \otimes_R C) & \xrightarrow{\bigoplus_{i\in I} j_i \otimes_R C} & \bigoplus_{i\in I}(A_i \otimes_R C) \end{array}$$

it follows that $\bigoplus_{i\in I} j_i \otimes_R C$ is injective. Since $- \otimes_R C$ preserves direct sums, it is evident that $j \otimes_R C$ is injective. $\square$

**Example 3.3.** More generally, similar to the above example, since $- \otimes_R C$ preserves direct limits, the direct limit of any system of pure short exact sequences is also pure exact.

**Example 3.4.** For any family of right $R$-modules $\{B_i\}_{i\in I}$, the direct sum $\bigoplus_{i\in I} B_i$ is a pure submodule of the product $\prod_{i\in I} B_i$.

*Proof.* Let $C \in R$-Mod. Consider the inclusion map $\bigoplus_{i\in I} B_i \xrightarrow{j} \prod_{i\in I} B_i$. Define the map:

$$\varepsilon\colon \left(\prod_{i\in I} B_i\right) \otimes_R C \to \prod_{i\in I}(B_i \otimes_R C), \quad \varepsilon\left(\{b_i\}_{i\in I} \otimes_R c\right) = \{b_i \otimes_R c\}_{i\in I}.$$

Then the following diagram commutes:

$$\begin{array}{ccc} \left(\bigoplus_{i\in I} B_i\right) \otimes_R C & \xrightarrow{j\otimes_R C} & \left(\prod_{i\in I} B_i\right) \otimes_R C \\ \downarrow \cong & & \downarrow \varepsilon \\ \bigoplus_{i\in I}(B_i \otimes_R C) & \rightarrowtail & \prod_{i\in I}(B_i \otimes_R C) \end{array}$$

It is easily seen that $\varepsilon \circ j \otimes_R C$ is injective, and therefore $j \otimes_R C$ is injective. □

**Example 3.5.** Let $A \subseteq B \subseteq C$ be right $R$-modules. If $A \subseteq C$ is a pure submodule, then $A \subseteq B$ is also a pure submodule. Conversely, if $A \subseteq B$ is pure and $B \subseteq C$ is pure, then $A \subseteq C$ is pure.

**Example 3.6.** Let $\varphi\colon R \to S$ be a ring homomorphism. Then $S$ can be viewed as a left $R$-module via $\varphi$. If $\mathscr{E}$ is a pure exact sequence in Mod-$R$, then $\mathscr{E} \otimes_R S$ is pure in Mod-$S$.

**Observation 3.2.** Let $C \in$ Mod-$R$ and consider the projective resolution:

$$\mathscr{E}\colon 0 \to K \xrightarrow{\varphi} P \to C \to 0.$$

Let $X \in R$-Mod. Then $\mathscr{E} \otimes_R X$ is exact if and only if

$$\operatorname{Tor}_1^R(C, X) = \ker\left(\varphi \otimes_R X\colon K \otimes_R X \to P \otimes_R X\right) = 0.$$

**Theorem 3.1** (Characterization of Flat Modules)**.** A right $R$-module $C$ is flat if and only if every short exact sequence

$$\mathscr{E}\colon 0 \to A \to B \to C \to 0$$

in Mod-$R$ is pure.

*Proof.* First, assume that $C$ is flat. Therefore, $\operatorname{Tor}_n^R(C, X) = 0$ for all $n \in \mathbb{N}$ and $X \in R$-Mod. By the long exact sequence in homology, it follows that every short exact sequence $\mathscr{E}\colon 0 \to A \to B \to C \to 0$ is pure.

$$\begin{array}{ccccccc}
0 & \leftarrow & C \otimes X & \leftarrow & B \otimes X & \leftarrow & A \otimes X \\
& & & \delta^0 & & & \\
& & \operatorname{Tor}_1^R(C, X) = 0 & \leftarrow & \operatorname{Tor}_1^R(B, X) & \leftarrow & \operatorname{Tor}_1^R(A, X) \\
& & & \delta^1 & & & \\
& & \operatorname{Tor}_2^R(C, X) & \leftarrow & \cdots\cdots & & \cdots\cdots
\end{array}$$

The converse follows immediately from Observation 3.2. □

**Corollary 3.3.** Let $\mathscr{E}\colon 0 \to A \to B \to C \to 0$ be an exact sequence in Mod-$R$.

(a) If $B$ is flat, then $\mathscr{E}$ is pure if and only if $C$ is flat.

(b) If $C$ is flat, then $B$ is flat if and only if $A$ is flat.

*Proof.* (a) Let $X \in R$-Mod. Applying the long exact sequence in homology to the short exact sequence $\mathscr{E}\colon 0 \to A \to B \to C \to 0$, we obtain:

$$\begin{array}{ccccccc} 0 & \longleftarrow & C \otimes X & \longleftarrow & B \otimes X & \longleftarrow & A \otimes X \\ & & & \delta^0 & & & \\ & & \mathrm{Tor}_1^R(C,X) & \longleftarrow & \mathrm{Tor}_1^R(B,X) = 0 & \longleftarrow & \mathrm{Tor}_1^R(A,X) \\ & & & \delta^1 & & & \\ & & \mathrm{Tor}_2^R(C,X) & \longleftarrow & \cdots\cdots & & \cdots\cdots \end{array}$$

From the diagram, it follows immediately that $\mathscr{E}$ is pure if and only if

$$\mathrm{Tor}_1^R(C,X) = 0$$

for each $X \in R$-Mod, which is equivalent to $C$ being flat.

(b) If $C$ is flat, then $\mathrm{Tor}_n^R(C,X) = 0$ for all $n \in \mathbb{N}$ and $X \in R$-Mod. By the long exact sequence in homology:

$$\begin{array}{ccccccc} 0 & \longleftarrow & C \otimes X & \longleftarrow & B \otimes X & \longleftarrow & A \otimes X \\ & & & \delta^0 & & & \\ & & \mathrm{Tor}_1^R(C,X) = 0 & \longleftarrow & \mathrm{Tor}_1^R(B,X) & \longleftarrow & \mathrm{Tor}_1^R(A,X) \\ & & & \delta^1 & & & \\ & & \mathrm{Tor}_2^R(C,X) = 0 & \longleftarrow & \cdots\cdots & & \cdots\cdots \end{array}$$

From the diagram, we obtain

$$\mathrm{Tor}_n^R(B,X) \cong \mathrm{Tor}_n^R(A,X)$$

for each $X \in R$-Mod. Thus, $B$ is flat if and only if $A$ is flat.

□

**Definition 3.3.** A module $P_R$ is said to be **finitely presented** (f.p.) if there exists a short exact sequence

$$0 \to K \to F \to P \to 0$$

in Mod-$R$, where $K$ is finitely generated (f.g.) and $F$ is a free module of finite rank. Equivalently, there exists an exact sequence in Mod-$R$ of the form:

$$R^m \to R^n \to P \to 0.$$

**Theorem 3.2** (Characterization of Pure Exact Sequences)**.** For any short exact sequence $\mathscr{E}: 0 \to A \to B \to C \to 0$ in Mod-$R$, the following are equivalent:

(a) $\mathscr{E}$ is pure exact.

(b) $\mathscr{E} \otimes_R C'$ is exact for any finitely presented (f.p.) left $R$-module $C'$.

(c) If $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_m \in B$, and $s_{ij} \in R$ ($1 \leq i \leq m$, $1 \leq j \leq n$) are given such that

$$a_j = \sum_i b_i s_{ij},$$

then there exist $a'_1, \ldots, a'_m \in A$ such that

$$a_j = \sum_i a'_i s_{ij}.$$

(d) Given any commutative diagram in Mod-$R$:

$$\begin{array}{ccc} R^n & \xrightarrow{\sigma} & R^m \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A & \hookrightarrow & B \end{array}$$

there exists $\vartheta \in \mathrm{Hom}_R(R^m, A)$ such that $\vartheta\sigma = \alpha$.

$$\begin{array}{ccc} R^n & \xrightarrow{\sigma} & R^m \\ \downarrow{\scriptstyle \alpha} & \swarrow{\scriptstyle \vartheta} & \downarrow{\scriptstyle \beta} \\ A & \hookrightarrow & B \end{array}$$

*Proof.* • (a) $\to$ (b) The desired result implied immediately from definition of pure exact sequence.

• (b) $\to$ (c) We consider the submodule of $R^n = \bigoplus_{j=1}^n Re_j$

$$K = \left\langle \left\{ \sum_{j=1}^m s_{ij} e_j \mid 1 \leq i \leq m \right\} \right\rangle$$

Thus the left $R$ - module $R^n/K$ is f.p. and therefore $A \otimes_R (R^n/K) \xrightarrow{\rho} B \otimes_R (R^n/K)$ is injective. Since the sequence is exact

$$A \otimes_R K \to A \otimes (R^n) \to A \otimes (R^n/K) \to 0$$

we can identify

$$A \otimes (R^n/K) \equiv A \otimes (R^n) / \mathrm{Im}(A \otimes_R K)$$

Similarly, we can identify

$$B \otimes (R^n/K) \equiv B \otimes (R^n) / \mathrm{Im}(B \otimes_R K)$$

Then

$$\rho\left(\left[\sum_j a_j \otimes e_j\right]\right) = \sum_j \sum_i [b_i s_{ij} \otimes e_j] = \sum_i \left[b_i \otimes \left(\sum_j s_{ij} e_j\right)\right] = 0$$

Since $\rho$ is injective, then $\sum_j a_j \otimes e_j \in \mathrm{Im}(A \otimes_R K)$. Equivalenlty, there are $a'_1, \ldots, a'_m \in A$ s.t.

$$\sum_j a_j \otimes e_j = \sum_i a'_i \otimes \left(\sum_j s_{ij} e_j\right) = \sum_j \left(\sum_i a'_i s_{ij}\right) \otimes e_j$$

Since $A \otimes_R -$ preserves direct sums, then $A \otimes_R R^n \equiv \bigoplus_{j=1}^n A \otimes Re_j$. Therefore, for each $j \in \{1, \ldots, n\}$, we have that

$$a_j = \sum_i a'_i s_{ij}$$

- (c) $\to$ (d) We denote $R^n = \bigoplus_{j=1}^n Re_j$ and $R^m = \bigoplus_{i=1}^m R\tilde{e}_i$. Then, for each $1 \le i \le m$ and $1 \le j \le n$ we set $a_j = \alpha(e_j)$ and $b_i = \beta(\tilde{e}_i)$. Since $\sigma(e_j) \in R^m$, then there are $s_{ij} \in R$ s.t.

$$\sigma(e_j) = \sum_i \tilde{e}_i s_{ij}$$

Then

$$a_j = \alpha(e_j) = \beta\sigma(e_j) = \sum_i b_i s_{ij}$$

By (c), there are $a'_1, \ldots, a'_m \in A$ s.t.

$$a_j = \sum_i a'_i s_{ij}$$

We define $\vartheta \in \mathrm{Hom}_R(R^m, A)$, where $\vartheta(\tilde{e}_i) = a'_i$. Then $\vartheta\sigma = \alpha$, as desired.

- (d) $\to$ (e) Let $M$ in Mod-$R$ a f.p. module Then exists exact sequence

$$R^n \xrightarrow{\sigma} R^m \xrightarrow{\tau} M \to 0$$

We will show that exactness of $\mathrm{Hom}_R\,(M, \mathscr{E})$, showing that

$$\psi_*\colon \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$$

is surjective. Let $\gamma \in \mathrm{Hom}_R(M, C)$. By freeness of $R^m$, exists $\beta \in \mathrm{Hom}_R\,(R^m, B)$ s.t. the following diagram commutes

$$\begin{array}{ccccc} R^m & \xrightarrow{\tau} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ B & \xrightarrow{\psi} & C & \longrightarrow & 0 \end{array}$$

We notice that

$$\psi\beta\sigma = \gamma\tau\sigma = 0$$

and therefore $\mathrm{Im}(\beta\sigma) \subseteq \ker\psi = A$. We set $\alpha = \beta\sigma$. Then the following diagram commutes

$$\begin{array}{ccccccc} R^n & \xrightarrow{\sigma} & R^m & \xrightarrow{\tau} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ A & \hookrightarrow & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \end{array}$$

By (d), there exists $\vartheta \in \mathrm{Hom}_R\,(R^m, A)$ s.t.

$$\alpha = \vartheta\sigma \Rightarrow \beta\sigma = \vartheta\sigma \Rightarrow (\beta - \vartheta)\,\sigma = 0$$

From the above relation, since $(M, \tau)$ is the cokernel of $\sigma$, then exists $\lambda \in \mathrm{Hom}_R\,(M, B)$ s.t. the following diagram commutes

$$\begin{array}{ccccc} R^n & \xrightarrow{\sigma} & R^m & \xrightarrow{\tau} & M \\ & & {\scriptstyle\beta-\vartheta}\downarrow & \swarrow{\scriptstyle\lambda} & \\ & & B & & \end{array}$$

Therefore

$$\psi\lambda\tau = \psi(\beta - \vartheta) = \psi\beta - \psi\vartheta = \gamma\tau$$

Since $\tau$ is right invertible then $\psi\lambda = \gamma$, as we desire.

- (e) $\to$ (f) It's true that every module is direct limit of f.p. modules. For the proof of this statement we refer the reader to [2] (Lazard, Govorov Theorem). Therefore $C$ is the direct limit of some direct system $(C_i, \gamma_i)_{i\in I}$, where $C_i$'s are f.p. modules.

- We consider the pullback diagram of $\gamma_i, \psi$

$$\begin{array}{ccccc} B_i & \xrightarrow{\psi_i} & C_i & \longrightarrow & 0 \\ \downarrow{\beta_i} & & \downarrow{\gamma_i} & & \\ B & \xrightarrow{\psi} & C & \longrightarrow & 0 \end{array}$$

where $B_i = \{(x,y) \in C_i \oplus B \mid \gamma_i(x) = \psi(y)\}$ and $\psi_i = \pi_1,\ \beta_i = \pi_2$.

- By Lemma 1.3,the induced following diagram commutes

$$\begin{array}{lccccccccc} \mathscr{E}_i: & 0 & \longrightarrow & A & \longrightarrow & B_i & \xrightarrow{\psi_i} & C_i & \longrightarrow & 0 \\ & & & \| & & \downarrow{\beta_i} & & \downarrow{\gamma_i} & & \\ \mathscr{E} & 0 & \longrightarrow & A & \hookrightarrow & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \end{array}$$

- By (e), there exists $\lambda_i \in \mathrm{Hom}_R(C_i, B)$ s.t. $\psi\lambda_i = \gamma_i$. Let $\rho_i \colon C_i \to B_i$ defined by $\rho_i(x) = (x, \lambda_i(x))$. Then it's obvious that $\psi_i\rho_i = \mathrm{id}_{C_i}$ and therefore $\mathscr{E}_i$ splits. I can be easily seen that $\mathscr{E} = \varinjlim \mathscr{E}_i$.

- (f) $\to$ (a) The result implied immediately by Example 3.3.

□

**Corollary 3.4.** Let $M$ be an $R$-module and $S \subseteq M$ a submodule. Then $S$ is pure if and only if for every $m \geq 1$, for every finitely generated submodule $T \subseteq R^m$, and for every $h \colon T \to S$, whenever $h$ can be extended to $\tilde{h} \colon R^m \to M$, it can also be extended to a $\vartheta \colon R^m \to S$, so that the following diagram commutes:

$$\begin{array}{ccc} T & \hookrightarrow & R^m \\ {\scriptstyle h}\downarrow & \swarrow{\scriptstyle \vartheta} & \downarrow{\scriptstyle \tilde{h}} \\ S & \hookrightarrow & M \end{array}$$

*Proof.* We assume that $S$ is pure. Since $T$ is f.g., there exists an epimorphism, for some $n \in \mathbb{N}$,

$$R^n \xrightarrow{\pi} T.$$

We consider a morphism $h \colon T \to S$ such that it can be extended to $\tilde{h} \colon R^m \to T$. Therefore, the following diagram commutes:

$$\begin{array}{ccc} T & \overset{i}{\hookrightarrow} & R^m \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \tilde{h}} \\ S & \underset{j}{\hookrightarrow} & M \end{array}$$

The existence of the morphism $\pi$ implies the following commuting diagram:

$$\begin{array}{ccc} R^n & \xrightarrow{i\pi} & R^m \\ {\scriptstyle h\pi}\downarrow & & \downarrow{\scriptstyle \tilde{h}} \\ S & \underset{j}{\hookrightarrow} & M \end{array}$$

and from Theorem 3.2, there exists $\varphi\colon R^m \to S$ such that the following diagram commutes:

$$\begin{array}{ccc} R^n & \xrightarrow{i\pi} & R^m \\ {\scriptstyle h\pi}\downarrow & \swarrow{\scriptstyle \vartheta} & \downarrow{\scriptstyle \tilde{h}} \\ S & \underset{j}{\hookrightarrow} & M \end{array}$$

Thus,

$$\vartheta i\pi = h\pi \Rightarrow \vartheta i = h,$$

so $\vartheta$ is the desired extension.

Conversely, consider a commutative diagram:

$$\begin{array}{ccc} R^n & \xrightarrow{f} & R^m \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \tilde{h}} \\ S & \underset{j}{\hookrightarrow} & M \end{array}$$

and we will show that there exists $\vartheta\colon R^m \to S$ such that the following diagram commutes:

$$\begin{array}{ccc} R^n & \xrightarrow{f} & R^m \\ {\scriptstyle h}\downarrow & \swarrow{\scriptstyle \vartheta} & \downarrow{\scriptstyle \tilde{h}} \\ S & \underset{j}{\hookrightarrow} & M \end{array}$$

If $T = \mathrm{Im} f \subseteq R^m$, then $T$ is f.g., and there is a commutative diagram:

$$\begin{array}{ccccc} R^n & \xrightarrow{f} & T & \overset{i}{\hookrightarrow} & R^m \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \tilde{h}i} & & \downarrow{\scriptstyle \tilde{h}} \\ S & = & S & \underset{j}{\hookrightarrow} & M \end{array}$$

By our assumptions, there exists $\vartheta\colon R^m \to S$ such that the following diagram commutes:

$$\begin{array}{ccccc} R^n & \xrightarrow{f} & T & \overset{i}{\hookrightarrow} & R^m \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \tilde{h}i} & \swarrow{\scriptstyle \vartheta} & \downarrow{\scriptstyle \tilde{h}} \\ S & = & S & \underset{j}{\hookrightarrow} & M \end{array}$$

From the above diagram, it is easily seen that $\vartheta f = h$. $\square$

**Corollary 3.5.** Let $S \subseteq M$ be a submodule. Let

$$P = \bigoplus_{(T,R^m)} R^m \quad \text{and} \quad U = \bigoplus_{(T,R^m)} T$$

summed over the set of all $(T, R^m)$ where $m \geq 1$ and $T$ is finitely generated (f.g.).

Then $S$ is pure if and only if for any homomorphism $h\colon U \to S$, which can be extended to a homomorphism $\tilde{h}\colon P \to M$, it can also be extended to a homomorphism

$$\tilde{h}\colon P \to S$$

*Proof.* Use Corollary 3.4. □

# CHAPTER 4

# VANISHING OF Ext FUNCTOR

## 4.1 A Review of Ordinal Numbers

Consider the class WO of all well-ordered sets. If we denote by $\cong$ the relation "being isomorphic to" between ordered structures, then $\cong$ defines an equivalence relation on WO. An ordinal can be thought of as an equivalence class of WO under the relation $\cong$. More precisely, the class Ord of all ordinals satisfies the property that, for any well-ordered set $A$, there exists exactly one ordinal isomorphic to $A$.

**Observation 4.1.** If $A$ and $B$ are ordered sets, $A \hookrightarrow B$ means that $A$ is embeddable into $B$, i.e., there exists an order-preserving injective map from $A$ to $B$.

**Theorem 4.1** (Transfinite Induction)**.** Let $(A, <)$ be a well-ordered set and $P(x)$ a property defined on $A$ satisfying:

$$\forall a \in A,\ [(\forall b < a\, P(b)) \Rightarrow P(a)]$$

Then $P(a)$ is true for every $a \in A$.

*Proof.*

- Consider $B := \{a \in A \mid\ P(a) \text{ is not true}\}$. For the sake of contradiction, we assume that $B \neq \emptyset$.
- Since $A$ is well-ordered, we can consider $a = \min(B)$. Then $P(b)$ is true for every $b < a$, but $P(a)$ is false, which contradicts the hypothesis of the theorem. Thus, $B = \emptyset$.

□

**Definition 4.1** (Initial Segment). Let $(A, <)$ be a well-ordered set and $a \in A$. The **initial segment** of $A$ determined by $a$ is the subset of $A$ of the form $A_a = \{b \in A \mid b \leq a\}$.

**Proposition 4.1.** Let $(A, <)$ be a well-ordered set. If $B$ is a proper initial segment of $A$, then there is no embedding $f\colon A \to B$. In particular, $A$ and $B$ are not isomorphic.

*Proof.* For the sake of contradiction, we assume that there is an embedding $f\colon A \to B$.

- We shall prove by induction on $A$ that for all $x \in A$, it holds that $f(x) \geq x$. Let $a \in A$ and assume that for all $b < a$, $f(b) \geq b$. Let $b \in A$ such that $b < a$. Since $f$ preserves the order, we have $f(b) < f(a)$, and by the induction hypothesis, we also have $b \leq f(b) \geq b$, hence $b < f(a)$. The latter relation implies that $f(a) > b$ for all $b < a$, hence $f(a) \geq a$.
- Since $B$ is a proper subset of $A$, there exists $a \in A \setminus B$, and since $B$ is an initial segment of $A$, we then have $a > b$ for all $b \in B$. In particular, we have $a > f(a)$, hence a contradiction.

□

**Definition 4.2.** A set $A$ is called **transitive** if every element of $A$ is also a subset of $A$. Equivalently, $A$ is transitive if and only if for each $a \in A$ and $x \in a$, then $x \in A$.

**Lemma 4.1.** Let $A$ be a transitive set. Then $\in$ is a transitive relation on $A$ if and only if for every $a \in A$, $a$ is a transitive set.

*Proof.*
- First, we assume that $\in$ is transitive. Let $a \in A$. We want to prove that $a$ is a transitive set. Let $y \in a$ and $x \in y$. We want to show that $x \in a$. Since $\in$ is a transitive relation, it suffices to show that $x, y \in A$. Since $a \in A$ and $A$ is transitive, then $y \in A$ and therefore $y \subseteq A$. Thus, $x \in A$, and we're done.
- Conversely, assume that $a$ is a transitive set for all $a \in A$. Let $a, b, c \in A$ such that $a \in b \in c$. Since $c$ is a transitive set, this relation implies $a \in c$.

□

**Lemma 4.2.** A union of transitive sets is a transitive set.

*Proof.* Let $\{A_i\}_{i\in I}$ be a family of transitive sets and set $A := \bigcup_{i\in I} A_i$. We want to show that $A$ is transitive. Let $a \in A$ and $x \in a$. There exists $i \in I$ such that $a \in A_i$. Since $A_i$ is a transitive set, the relation $x \in a \in A_i$ implies $x \in A_i$, hence $x \in A$. □

**Definition 4.3.** A set $\alpha$ is called an **ordinal** if it is transitive and the pair $(\alpha, \in)$ is a well-ordered set.

**Remark 1.**

- The class Ord of all ordinals is not a set in the sense of axiomatic set theory.
- The definition above implies, in particular, that $\in$ is a well-order on $\alpha$, so it is a transitive relation. According to Lemma 4.1, this means that any element of $\alpha$ is a transitive set.

**Example 4.1.** Each natural number $n + 1 = \{0, \ldots, n-1\} \cup \{n\}$ is an ordinal. In addition, $\omega = \bigcup_{n\in\mathbb{N}} n$ is an ordinal.

**Definition 4.4.** Let $\mathscr{R}$ be a relation on a set $S$. Then $\mathscr{R}$ is a **strict ordering** (on $S$) if and only if $\mathscr{R}$ satisfies the strict ordering axioms:

(a) *Asymmetry*:
$$\forall a, b \in S : (a\mathscr{R}b) \implies \neg(b\mathscr{R}a)$$

(b) *Transitivity*:
$$\forall a, b, c \in S : (a\mathscr{R}b) \wedge (b\mathscr{R}c) \implies a\mathscr{R}c$$

**Proposition 4.2.** The binary relation $\in$ defines a strict order on Ord.

*Proof.*

- $\in$ *is transitive*: Let $\alpha \in \beta \in \gamma$, all in Ord. Since $\gamma$ is a transitive set, we have $\alpha \in \gamma$.
- $\in$ *is antisymmetric*: Assume there exist $\alpha, \beta \in$ Ord such that $\beta \in \alpha \in \beta$.
- Since $\beta$ is a transitive set, we have $\beta \in \beta$, and since $(\beta, \in)$ is well-ordered, this is a contradiction.

□

**Remark 2.** The order we consider on Ord will always be the one given by $\in$; thus, if $\alpha, \beta$ are ordinals, $\alpha < \beta$ means $\alpha \in \beta$.

**Proposition 4.3.** Let $\alpha$ be an ordinal. Then

$$\alpha = \{\beta \mid \beta \text{ is an ordinal and } \beta < \alpha\}$$

*Proof.* Let $\beta \in \alpha$, we shall show that $\beta$ is an ordinal.

- By Remark 1, we know that $\beta$ is a transitive set.
- Since $\alpha$ is a transitive set, we have $\beta \subseteq \alpha$, so the relation $\in$ defined on $\beta$ is the restriction of the relation $\in$ defined on $\alpha$. Since $(\alpha, \in)$ is well-ordered, this implies that $(\beta, \in)$ is well-ordered. Thus, $\beta$ is an ordinal.

□

**Corollary 4.1.** Let $\alpha, \beta \in \mathrm{Ord}$. Then

(a) $\alpha \subseteq \beta$ if and only if for all $\delta \in \mathrm{Ord}$ : $\delta < \alpha \Rightarrow \delta < \beta$.

(b) $\alpha = \beta$ if and only if for all $\delta \in \mathrm{Ord}$ : $\delta < \alpha \Leftrightarrow \delta < \beta$.

**Corollary 4.2.** Let $\alpha, \beta \in \mathrm{Ord}$ such that $\alpha < \beta$. Then $\alpha$ is a proper initial segment of $\beta$.

**Lemma 4.3.** Let $\alpha, \beta$ be ordinals such that $\beta \not< \alpha$. Then $\gamma = \min(\beta \setminus \alpha)$ exists and is included in $\alpha$. If moreover $\alpha \subsetneq \beta$, then $\gamma = \alpha$, and so $\alpha \in \beta$.

*Proof.*

- The existence of $\gamma$ follows from the fact that $\beta \setminus \alpha \neq \emptyset$ and that $\beta$ is well-ordered.
- Note that since $\gamma \in \beta$, $\gamma$ is an ordinal and $\gamma < \beta$. Let $\delta$ be an ordinal such that $\delta < \gamma$. Since $\gamma < \beta$, we have $\delta \in \beta$. However, since $\delta < \gamma$, by the minimality of $\gamma$, we have $\delta \in \alpha$. This proves that $\gamma \subseteq \alpha$.
- Now assume that $\alpha \subset \beta$ and let $\delta < \alpha$; we also have $\delta \in \beta$. If $\delta > \gamma$, we would have $\alpha > \gamma$, i.e., $\gamma \in \alpha$, which by the definition of $\gamma$ is impossible. Since $\delta, \gamma \in \beta$ and $\beta$ is totally ordered, this implies $\delta < \gamma$. This proves that $\alpha \subseteq \gamma$, hence $\gamma = \alpha$.

□

**Lemma 4.4.** Let $\alpha, \beta$ be ordinals. Then $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$.

*Proof.*
- If $\alpha = \beta$, there is nothing to prove. If $\alpha < \beta$, the fact that $\beta$ is a transitive set implies $\alpha \subseteq \beta$.
- We assume that $\alpha \subset \beta$. In that case, Lemma 4.3 implies that $\alpha \in \beta$, and we're done.

□

**Proposition 4.4.** The order $<$ (which is also $\in$) is a total order on Ord.

*Proof.*
- Let $\alpha, \beta$ be ordinals such that $\beta \not< \alpha$. By Lemma 4.4, we have $\beta \not\subseteq \alpha$, which by Lemma 4.3 implies $\gamma = \min(\beta \setminus \alpha) \subseteq \alpha$.
- By Lemma 4.4, we have $\gamma \leq \alpha$. However, by the definition of $\gamma$, we cannot have $\gamma \in \alpha$, hence $\gamma = \alpha$, which implies $\alpha \in \beta$.

□

**Proposition 4.5.** If $\alpha \neq \beta$, then $\alpha$ and $\beta$ are not isomorphic.

*Proof.* Since $<$ is a total order, we can assume $\alpha < \beta$. Then $\alpha$ is a proper initial segment of $\beta$, which, by Proposition 4.1, implies that $\alpha$ and $\beta$ are not isomorphic. □

**Proposition 4.6.** The pair $(\mathrm{Ord}, <)$ is well-ordered.

*Proof.* Since the order is total, we only need to show that there is no strictly decreasing infinite sequence of ordinals:

$$\alpha_0 > \alpha_1 > \alpha_2 > \cdots > \alpha_n > \cdots$$

If such a sequence existed, then $\alpha_n \in \alpha_0$ for every $n \geq 0$, so $(\alpha_n)_{n>0}$ would be an infinite decreasing sequence of elements of $\alpha_0$, which would contradict the fact that $\alpha_0$ is well-ordered. □

**Proposition 4.7.**
(a) If $\alpha$ is an ordinal, then so is $\alpha \cup \{\alpha\}$. The ordinal $\alpha + 1 := \alpha \cup \{\alpha\}$ is called the **successor** of $\alpha$.

(b) If $A$ is a set of ordinals, then $\sup(A) = \bigcup A$ is an ordinal.

*Proof.* For the second part: Set $\delta = \bigcup A$. Then $\delta$ is a union of transitive sets, so by Lemma 4.2, it is a transitive set. To show that $\delta$ is well-ordered, note that $\delta \subset \mathrm{Ord}$ and that Ord is well-ordered. We will show that $\delta$ is the supremum of $A$. Clearly, $\alpha < \delta$ for any $\alpha \in A$. Let $\gamma \in \mathrm{Ord}$ be such that $\gamma > \alpha$ for all $\alpha \in A$. Let $\beta \in \delta$; then there exists $\alpha \in A$ such that $\beta \in \alpha < \gamma$, hence $\beta \in \gamma$. This proves that $\delta \subseteq \gamma$, hence $\delta \leq \gamma$. □

**Remark 3.**

- The definition of the successor of an ordinal is consistent with the usual definition of the successor of an integer: indeed, if $n \in \omega$, then $n + 1 = \{0, 1, \ldots, n\} = n \cup \{n\}$.
- $\alpha + 1$ is the smallest ordinal strictly greater than $\alpha$.
- $\sup(A)$ is not necessarily a maximum: take $A := \{2n \mid n \in \omega\}$, then $\sup(A) = \omega$, but $A$ has no maximum.
- However, if we take $A = \{0, 1, 3\}$, then $\sup(A) = \max(A) = 3$.
- If $\alpha$ is an ordinal, then, in particular, it is a set of ordinals, and in that case, we have $\sup \alpha = \alpha$.

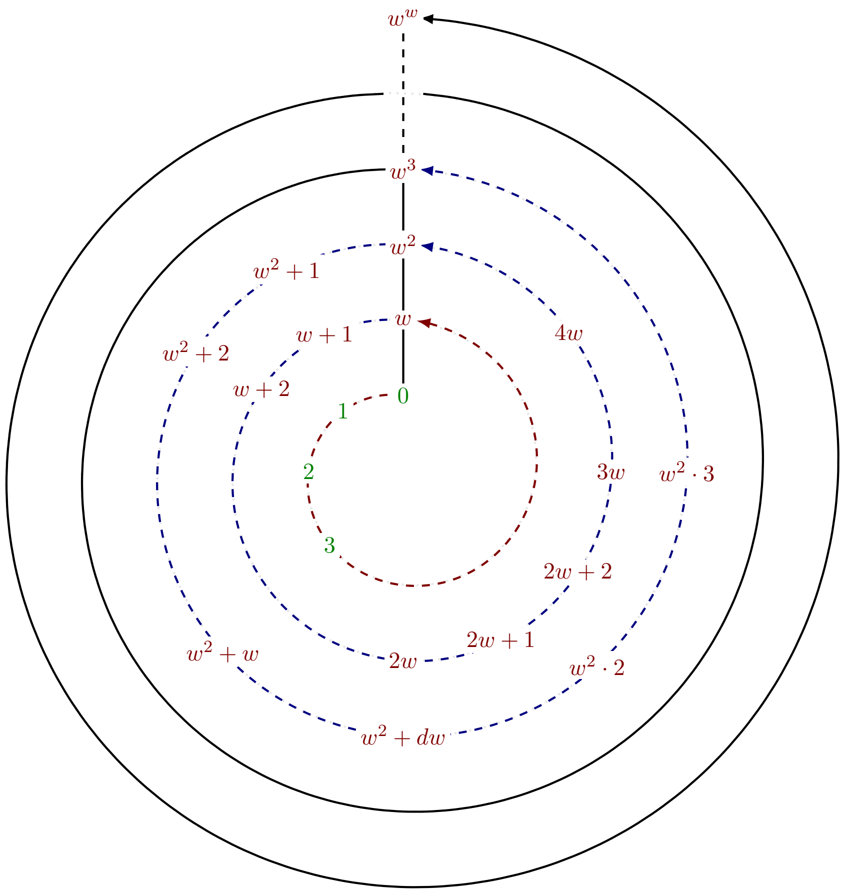
**Definition 4.5.** An ordinal that is neither a successor nor 0 is called a **limit ordinal** .

**Example 4.2.** $\omega$ is a limit ordinal (it is actually the smallest one).

**Corollary 4.3.**

- Thus, we can say that there are three kinds of ordinals: $0$, successor ordinals, and limit ordinals.
- The distinction between limit and successor ordinals is an important one since they have different properties. For example, a successor ordinal has a maximum, but a limit ordinal does not. We will also see that we usually separate the cases of successor and limit ordinals when making a proof by induction on ordinals.
- Proposition 4.7 gives us the tools to inductively construct ordinals. Remember that natural numbers are constructed by starting with $0$ and then repeatedly applying the successor map: we define $1$ as the successor of $0$, $2$ as the successor of $1$, and so on. Ordinals are constructed by alternately applying these two operations:
  - Taking the successor of the last ordinal defined.

- Once the successor operation has been repeated $\omega$ times, take the supremum of all the already defined ordinals.
- More precisely, we start by defining $0$, then apply the successor operation $\omega$ times to construct the set of natural numbers. We then define $\omega$ as the supremum of all natural numbers. We then repeat the same process: after $\omega$ comes its successor $\omega+1 := \omega \cup \{\omega\}$, then $\omega+2 := (\omega+1)\cup\{\omega+1\}$, and so on. After applying the successor operation $\omega$ times, we arrive at $\omega + \omega :=$ sup$\{n \in \omega\}(\omega + n)$.
- By repeating this process indefinitely, we construct the class of ordinals.




We conclude our review of ordinal numbers with the most important result. This is a generalization of ordinary induction and is a very useful tool for applying induction to continuous chains of sets in which the index set is a set of ordinals.

**Theorem 4.2** (Transfinite induction on Ord)**.** Let $\mathscr{P}(x)$ be a property defined on ordinals such that:

- $\mathscr{P}(0)$ is true.
- If $\mathscr{P}(\alpha)$ is true, then $\mathscr{P}(\alpha+1)$ is true.
- If $\alpha$ is a limit ordinal and $\mathscr{P}(\beta)$ is true for every $\beta < \alpha$, then $\mathscr{P}(\alpha)$ is true.

Then $\mathscr{P}(\alpha)$ is true for every $\alpha \in \mathrm{Ord}$.

**Theorem 4.3** (Transfinite induction on an ordinal)**.** Let $\alpha \in \mathrm{Ord}$ and let $\mathscr{P}(x)$ be a property defined on $\alpha$ such that:

- $\mathscr{P}(0)$ is true.
- If $\beta+1 < \alpha$ and $\mathscr{P}(\beta)$ is true, then $\mathscr{P}(\beta+1)$ is true.
- If $\beta < \alpha$ is a limit ordinal and $\mathscr{P}(\gamma)$ is true for every $\gamma < \beta$, then $\mathscr{P}(\beta)$ is true.

Then $\mathscr{P}(\beta)$ is true for every $\beta < \alpha$.

## 4.2 Vanishing of Ext Functor

**Lemma 4.5.** Let $N$ be a module. Let $\{M_\alpha \mid \alpha < \kappa\}$ be a continuous chain of modules. Put $M = \bigcup_{\alpha<\kappa} M_\alpha$. We assume that:

- $\mathrm{Ext}^1_R(M_0, N) = 0$, and
- $\mathrm{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) = 0$ whenever $\alpha+1 < \kappa$.

Then $\mathrm{Ext}^1_R(M, N) = 0$.

*Proof.* Put $M = M_k$. By Theorem 4.3, we will prove the desired result using induction on $\alpha \leq \kappa$, that is, $\mathrm{Ext}^1_R(M_\alpha, N) = 0$ for each $\alpha \leq \kappa$.

(a) *Zero Case*. By assumption, it is true that $\mathrm{Ext}^1_R(M_\alpha, N) = 0$ for $\alpha = 0$.

(b) *Successor Case*. Let $\alpha = \beta+1 < \kappa$. We assume that $\mathrm{Ext}^1_R(M_\beta, N) = 0$. If we apply the functor $\mathrm{Ext}^1_R(-, N)$ to the short exact sequence

$$0 \to M_\beta \hookrightarrow M_\alpha \to M_\alpha/M_\beta \to 0,$$

then we obtain

$$0 = \mathrm{Ext}^1_R(M_\alpha/M_\beta, N) \to \mathrm{Ext}^1_R(M_\alpha, N) \to \mathrm{Ext}^1_R(M_\beta, N) = 0.$$

(c) *Limit Case*. Let $\alpha < \kappa$ be a limit ordinal. We consider the short exact sequence

$$0 \to N \to I \xrightarrow{\pi} I/N,$$

where $I$ is injective. We want to show that $\mathrm{Ext}^1_R(M_\alpha, N) = 0$, so it suffices to show that

$$\pi_* \colon \mathrm{Hom}(M_\alpha, I) \to \mathrm{Hom}(M_\alpha, I/N)$$

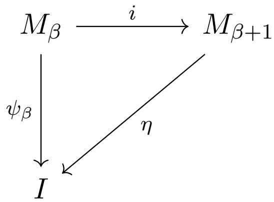
is surjective. Let $\varphi \in \mathrm{Hom}(M_\alpha, I/N)$. We seek $\psi \in \mathrm{Hom}(M_\alpha, I)$ such that

$$\pi\psi = \varphi$$

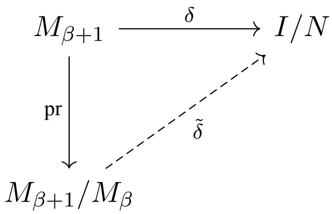
- We construct a continuous chain of homomorphisms $\{\psi_\beta \colon M_\beta \to I\}_{\beta<\alpha}$ such that

  $$\varphi \restriction_{M_\beta} = \pi\psi_\beta \quad \text{and} \quad \psi_\beta \restriction_{M_\gamma} = \psi_\gamma, \quad \forall \gamma < \beta < \alpha.$$

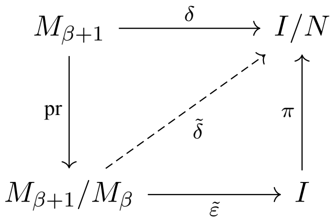
  Then, if we set $\psi := \bigcup_{\beta<\alpha} \psi_\beta$, we obtain $\psi \in \mathrm{Hom}(M_\alpha, I)$ and $\varphi = \pi\psi$.
- We construct this chain by induction on $\beta < \alpha$:
  - We assume that $\psi_\beta$ is already defined. Since $I$ is injective, there exists $\eta \in \mathrm{Hom}(M_{\beta+1}, I)$ that extends $\psi_\beta$:




  - If $\delta = \varphi \restriction_{M_{\beta+1}} - \pi\eta \in \mathrm{Hom}(M_{\beta+1}, I/N)$, it can be easily seen that $\delta \restriction_{M_\beta} = 0$. Hence, there exists $\tilde{\delta} \colon M_{\beta+1}/M_\beta \to I/N$ such that the following diagram commutes:




  Since $\mathrm{Ext}^1_R(M_{\beta+1}/M_\beta, N) = 0$, there exists $\tilde{\varepsilon} \colon M_{\beta+1}/M_\beta \to I$ such that $\pi\tilde{\varepsilon} = \tilde{\delta}$. We set $\varepsilon = \tilde{\varepsilon} \circ \mathrm{pr}$. Notice that $\varepsilon \restriction_{M_\beta} = 0$:



- Thus, $\pi\varepsilon = \delta = \varphi \restriction_{M_{\beta+1}} - \pi\eta$. Therefore, if we set $\psi_{\beta+1} = \varepsilon + \eta$, then $\psi_{\beta+1}$ satisfies the desired properties.
- For a limit ordinal $\beta < \alpha$, we put $\psi_\beta = \bigcup_{\gamma<\beta} \psi_\gamma$.

□

## 4.3 Sets of Modules and Complete Cotorsion Theories

**Remark 4.** In general, given a class of modules $\mathscr{S}$, we do not have specific criteria for testing whether the cotorsion theory cogenerated by $\mathscr{S}$ is complete or not. A useful application of the preceding lemma is that whenever $\mathscr{S}$ is a **set** of modules, the cotorsion theory cogenerated by $\mathscr{S}$ is complete.

**Lemma 4.6.** Let $\mathscr{S}$ be a set of modules. If $X = \bigoplus_{S\in\mathscr{S}} S$, then $X^\perp = S^\perp$.

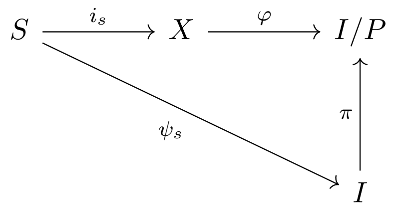
*Proof.* • Let $P \in \mathscr{S}^\perp$. We consider an injective resolution of $P$:

$$0 \to P \xrightarrow{i} I \xrightarrow{\varepsilon} I/P \to 0.$$

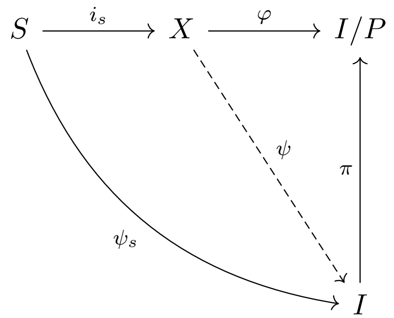
We want to show that $\operatorname{Ext}^1_R(X, P) = 0$, equivalently, we want to show that

$$\varepsilon_*\colon \operatorname{Hom}(X, I) \to \operatorname{Hom}(X, I/P)$$

is surjective. Let $\varphi\colon X \to I/P$. If we denote by $i_S$ the embedding of $S$ into $X$, then for each $S \in \mathscr{S}$, there exists $\psi_S \in \operatorname{Hom}(S, I)$ such that the following diagram commutes:




By the universal property of direct sums, there is a unique $\psi\colon X \to I$ such that $\psi_s = \psi \circ i_S$. It can be easily seen that $\varphi = \pi \circ \psi$.



- The converse relation follows immediately from the projection map $\pi_S \colon X \to S$.

$\square$

**Theorem 4.4.** Let $\mathscr{S}$ be a set of modules. Let $M$ be a module. There exists a short exact sequence

$$0 \to M \hookrightarrow P \to N \to 0,$$

where $P \in \mathscr{S}^\perp$, $N \in^\perp (\mathscr{S}^\perp)$, and $P$ is the union of a continuous chain of submodules such that:

- $P_0 = M$, and
- $P_{\alpha+1}/P_\alpha$ is isomorphic to a direct sum of copies of elements of $\mathscr{S}$ for each $\alpha + 1 < \lambda$.

In particular, $M \hookrightarrow P$ is a special $\mathscr{S}^\perp$-preenvelope of $M$.

*Proof.* By Lemma 4.6, w.l.o.g., we can assume that $\mathscr{S}$ consists of a single element $S$.

- Let $0 \to K \overset{\mu}{\to} F \to S \to 0$ be a short exact sequence with $F$ being a free module. Let $\lambda$ be an infinite cardinal such that $K$ is $< \lambda$-generated.
- We shall inductively construct a continuous chain of modules $\{P_\alpha \mid \alpha < \lambda\}$ that satisfies the assumptions of the theorem. We set $P_0 = M$.
- If $\alpha + 1 < \lambda$, we assume that we have already constructed $P_\beta$ for each $\beta \leq \alpha$. If $X_\alpha = \mathrm{Hom}(K, P_\alpha)$, we define

$$\mu_\alpha = \bigoplus_{X_\alpha} \mu \in \mathrm{Hom}\left(K^{(X_\alpha)}, F^{(X_\alpha)}\right),$$

  that is, $\mu_\alpha$ is the direct sum of $X_\alpha$-copies of $\mu$. From the definition of $\mu_\alpha$, it is obvious that the following short exact sequence is implied:

$$0 \to K^{(X_\alpha)} \xrightarrow{\mu_\alpha} F^{(X_\alpha)} \to S^{(X_\alpha)} \to 0.$$

  Therefore, $\mu_\alpha$ is a monomorphism, and coker $\mu_\alpha$ is isomorphic to a direct sum of copies of $S$.

- Let $\varphi_\alpha \in \operatorname{Hom}(K^{(X_\alpha)}, P_\alpha)$ be the canonical morphism, where

$$\varphi_\alpha\left(\{k_\eta\}_{\eta\in X_\alpha}\right) = \sum_{\eta\in X_\alpha} \eta(k_\eta).$$

For each $\eta \in X_\alpha$, we denote by $\nu_\eta$ and $\nu'_\eta$ the canonical embeddings:

$$\nu_\eta \colon K \to K^{(X_\alpha)} \quad \text{and} \quad \nu'_\eta \colon F \to F^{(X_\alpha)}.$$

Some trivial but at the same time important remarks are that $\eta = \varphi_\alpha \circ \nu_\eta$ and $\nu_{\eta'} \circ \mu = \mu_\alpha \circ \nu_\eta$.

- We consider the pushout diagram of $\mu_\alpha$ and $\varphi_\alpha$:

$$\begin{array}{ccc} K^{(X_\alpha)} & \xrightarrow{\ \mu_\alpha\ } & F^{(X_\alpha)} \\ {\scriptstyle\varphi_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\psi_\alpha} \\ P_\alpha & \xrightarrow{\ i\ } & P_{\alpha+1} \end{array}$$

By the dual result of Lemma 1.3, we have that $i$ is a monomorphism and $P_{\alpha+1}/P_\alpha$ is isomorphic to a direct sum of copies of $S$.

- If $\alpha \leq \lambda$ is a limit ordinal, we define $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$. We set $P = \bigcup_{\alpha<\lambda} P_\alpha$.

- First, we will prove that $P \in S^\perp$. Equivalently, it suffices to show that

$$\mu^* \colon \operatorname{Hom}(F, P) \to \operatorname{Hom}(K, P)$$

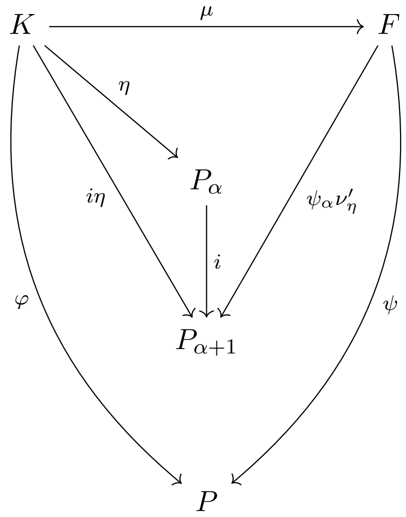
is surjective. Let $\varphi \in \operatorname{Hom}(K, P)$. Since $K$ is $<\lambda$-generated, there exists $\alpha < \lambda$ and $\eta \in X_\alpha$ such that $\eta(k) = \varphi(k)$ for each $k \in K$.

- If we combine the above relations, it follows that

$$\psi_\alpha \nu'_\eta \mu = \psi_\alpha \mu_\alpha \nu_\eta = i\varphi_\alpha \mu_\eta = i\eta.$$

If we define $\psi \colon F \to P$ such that $\psi(f) = \psi_\alpha \nu'_\eta(f)$ for each $f \in F$, then

$\varphi = \psi\mu$.




- It remains to show that $N = P/M \in^{\perp} \left(\mathscr{S}^{\perp}\right)$. Since $N = P/M$, we see that $N$ is the union of the continuous chain $\{N_\alpha \mid \alpha < \lambda\}$, where $N_\alpha = P_\alpha/M$. Let $X \in S^{\perp}$.
  - Since $P_0 = M$, we have $N_0 = 0$. Thus, it is obvious that $\mathrm{Ext}^1_R(N_0, X) = 0$.
  - If $\alpha + 1 < \lambda$, we have shown that $P_{\alpha+1}/P_\alpha$ is isomorphic to a direct sum of copies of $S$, and since $X \in S^{\perp}$, it follows that $\mathrm{Ext}^1_R\left(P_{\alpha+1}/P_\alpha, X\right) = 0$.

  By Lemma 4.5, we have that $\mathrm{Ext}^1_R(N, X) = 0$, and therefore $N \in^{\perp} \left(S^{\perp}\right)$.

□

**Corollary 4.4.** Let $\mathscr{S}$ be a set of modules. Then the cotorsion theory

$$\mathfrak{C}_{\mathscr{S}} = \left({}^{\perp}\left(\mathscr{S}^{\perp}\right), \mathscr{S}^{\perp}\right)$$

is complete.

# CHAPTER 5

# ALL MODULES HAVE FLAT COVERS

## 5.1 Existence of Special Precovers Implies Cover's Existence

We aim to prove that each module has a flat cover. A key point in proving this result is to show that the existence of a special flat precover induces the existence of a flat cover. If we combine this result with the main result of Chapter 5 (that the flat cotorsion theory is complete), we will have proved the desired result.

**Theorem 5.1.** Let $R$ be a ring and $M$ be a module. Let $\mathscr{C}$ be a class of modules closed under extensions and arbitrary direct limits. Assume that $M$ has a special $\mathscr{C}^{\perp}$-preenvelope $\varphi$, with coker$\varphi \in \mathscr{C}$. Then $M$ has a $\mathscr{C}^{\perp}$ envelope.

**Definition 5.1.** With the above assumptions, an exact sequence

$$0 \to M \to F \to C \to 0, \quad C \in \mathscr{C}$$

is called an **Ext-generator** if for each exact sequence

$$0 \to M \to F' \to C' \to 0, \quad C' \in \mathscr{C}$$

there exist $f \in \mathrm{Hom}_R(F', F)$ and $g \in \mathrm{Hom}_R(C', C)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

**Lemma 5.1.** With the assumptions of Theorem 5.1, assume that $0 \to M \to F \to C \to 0$ is an Ext-generator. Then there exists an Ext-generator $0 \to M \to F' \to C' \to 0$ and a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

such that $\ker f = \ker f'f$ in any commutative diagram whose rows are Ext-generators:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \\
0 & \longrightarrow & M & \longrightarrow & F'' & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

*Proof.* We assume, for the sake of contradiction, that the above result is not true. Then, for any Ext-generator $0 \to M \to F' \to C' \to 0$ and a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

there exists a commutative diagram whose rows are Ext-generators:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \\
0 & \longrightarrow & M & \longrightarrow & F'' & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

such that $\ker f \subsetneqq \ker(f'f)$. By induction, we will construct for each ordinal $\alpha$ a strictly increasing chain of submodules of $F$, $\{\ker f_{0\beta} \mid \beta < \alpha\}$, which leads to a contradiction.

- **Zero Case.** We set $F' = F_0 = F, C' = C_0 = C$, and $f = \mathrm{id}_F, g = \mathrm{id}_C$. Then there exist $F_1 = F''$ and $C_1 = C'' \in \mathscr{C}$ with a pair of morphisms $f_{01} = f'$, $g_{01} = g'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \\
0 & \longrightarrow & M & \longrightarrow & F'' & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

  its rows are Ext-generators, and $0 = \ker f \subsetneqq \ker f_{01}$.

- **Successor Case.** We assume that the Ext-generator

$$0 \to M \to F_\alpha \to C_\alpha \to 0$$

  is defined together with $f_{\beta\alpha} \in \mathrm{Hom}(F_\beta, F_\alpha)$ and $g_{\beta\alpha} \in \mathrm{Hom}(C_\beta, C_\alpha)$ such that for any $\beta \leq \alpha$, the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f_{0\beta}} & & \downarrow{\scriptstyle g_{0\beta}} & & \\
0 & \longrightarrow & M & \longrightarrow & F_\beta & \longrightarrow & C_\beta & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f_{\beta\alpha}} & & \downarrow{\scriptstyle g_{\beta\alpha}} & & \\
0 & \longrightarrow & M & \longrightarrow & F_\alpha & \longrightarrow & C_\alpha & \longrightarrow & 0
\end{array}
$$

and $\ker(f_{0\beta}) \subsetneq \ker(f_{0\beta'})$ for each $\beta \leq \beta' \leq \alpha$. Then there exists an Ext-generator $0 \to M \to F_{\alpha+1} \to C_{\alpha+1} \to 0$ and $f_{\alpha,\alpha+1}$ and $g_{\alpha,\alpha+1}$ such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f_{0\alpha}} & & \downarrow{\scriptstyle g_{0\alpha}} & & \\
0 & \longrightarrow & M & \longrightarrow & F_\alpha & \longrightarrow & C_\alpha & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f_{\alpha,\alpha+1}} & & \downarrow{\scriptstyle g_{\alpha,\alpha+1}} & & \\
0 & \longrightarrow & M & \longrightarrow & F_{\alpha+1} & \longrightarrow & C_{\alpha+1} & \longrightarrow & 0
\end{array}
$$

its rows are Ext-generators, and $\ker f_{0\alpha} \subsetneq \ker f_{0\alpha+1}$, where $f_{\beta,\alpha+1} = f_{\alpha,\alpha+1} f_{\beta\alpha}$ and $g_{\beta,\alpha+1} = g_{\alpha,\alpha+1} g_{\beta\alpha}$, for all $\beta \leq \alpha$.

- **Limit Case.** We assume that $\alpha$ is a limit ordinal and that

$$0 \to M \to F_\beta \to C_\beta \to 0$$

is defined for each $\beta < \alpha$ together with $f_{\beta,\beta'} \in \mathrm{Hom}_R(F_\beta, F_{\beta'})$ and $g_{\beta,\beta'} \in \mathrm{Hom}_R(C_\beta, C_{\beta'})$, whenever $\beta \leq \beta' < \alpha$. Then the "triad":

$$\Big(0 \to M \to F_\beta \to C_\beta \to 0, \big(f_{\beta\beta'}, g_{\beta,\beta'}\big)_{\beta \leq \beta' < \alpha}\Big)_{\beta < \alpha}$$

is a directed system. Let

$$(0 \to M \to F_\alpha \to C_\alpha \to 0, f_{\beta\alpha}, g_{\beta\alpha})_{\beta < \alpha}$$

be the direct limit of the above system, where $F_\alpha = \varinjlim F_\beta$ and $C_\alpha = \varinjlim C_\beta \in \mathscr{C}$, since $\mathscr{C}$ is closed under arbitrary direct limits. Then it can be easily seen that

$$0 \to M \to F_\alpha \to C_\alpha \to 0$$

is an Ext-generator and that $\ker(f_{0\beta}) \subsetneqq \ker(f_{0\alpha})$, for all $\beta < \alpha$.

□

**Lemma 5.2.** With the assumptions of Theorem 5.1, assume that $0 \to M \to F \to C \to 0$ is an Ext-generator. Then there exists an Ext-generator $0 \to M \to F' \to C' \to 0$ and a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow f & & \downarrow g & & \\
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

such that $\ker(f') = 0$ in any commutative diagram whose rows are Ext-generators:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \| & & \downarrow f' & & \downarrow g' & & \\
0 & \longrightarrow & M & \longrightarrow & F'' & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

*Proof.* We will inductively construct a countable system $\mathbb{D}$ of Ext-generators using Lemma 5.1, and we shall show that the direct limit of this system satisfies the desired property.

- **Zero Case.** We set $0 \to M \to F \to C \to 0$ as the 0-th term of $\mathbb{D}$. By Lemma 5.1, there exists an Ext-generator $0 \to M \to F_1 \to C_1 \to 0$ and a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow f_0^1 & & \downarrow g_0^1 & & \\
0 & \longrightarrow & M & \longrightarrow & F_1 & \longrightarrow & C_1 & \longrightarrow & 0
\end{array}$$

such that for any commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow f_0^1 & & \downarrow g_0^1 & & \\
0 & \longrightarrow & M & \longrightarrow & F_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow f' & & \downarrow g' & & \\
0 & \longrightarrow & M & \longrightarrow & F'' & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

whose rows are Ext-generators, then $\ker\left(f' f_0^1\right) = \ker f_0^1$.

- **Inductive Step.** We assume that for some $m \in \mathbb{N}$, we have constructed a directed system $\mathbb{D}_m$ of Ext-generators:

$$(0 \to M \to F_i \to C_i \to 0)_{i=1}^m$$

$$(f_{ij} \in \mathrm{Hom}(F_i, F_j),\ g_{ij} \in \mathrm{Hom}(C_i, C_j))_{i \leq j \leq m}$$

where $f_i^{i+1}$ and $g_i^{i+1}$ are defined as in the zero case, and for each $i \leq j$, we define

$$f_i^j = f_{j-1}^j \circ \cdots \circ f_i^{i+1} \quad \text{and} \quad g_i^j = g_{j-1}^j \circ \cdots \circ g_i^{i+1}.$$

By Lemma 5.1, there exists an Ext-generator $0 \to M \to F_{n+1} \to C_{n+1} \to 0$ and a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F_n & \longrightarrow & C_n & \longrightarrow & 0 \\
 & & \| & & \downarrow f_n^{n+1} & & \downarrow g_n^{n+1} & & \\
0 & \longrightarrow & M & \longrightarrow & F_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0
\end{array}$$

such that for any commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F_n & \longrightarrow & C_n & \longrightarrow & 0 \\
 & & \| & & \downarrow f_n^{n+1} & & \downarrow g_n^{n+1} & & \\
0 & \longrightarrow & M & \longrightarrow & F_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0 \\
 & & \| & & \downarrow f' & & \downarrow g' & & \\
0 & \longrightarrow & M & \longrightarrow & F'' & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

So we have constructed a countable direct system $\mathbb{D}$ of Ext-generators:

$$(0 \to M \to F_n \to C_n \to 0)_{n \in \mathbb{N}}$$

$$(f_{ij} \in \mathrm{Hom}(F_i, F_j),\ g_{ij} \in \mathrm{Hom}(C_i, C_j))_{i \leq j}$$

with the above properties. We consider the direct limit of $\mathbb{D}$:

$$\left(0 \to M \to F' \to C' \to 0, (\varphi_n, \psi_n)_{n\in\mathbb{N}}\right)$$

so that $F' = \varinjlim F_n$ and $C' = \varinjlim C_n \in \mathscr{C}$. We will show that this direct limit satisfies the desired property.

We consider a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \\
0 & \longrightarrow & M & \longrightarrow & F'' & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

whose rows are Ext-generators. We assume, for the sake of contradiction, that there exists $[x] \in \ker(f')$ and $[x] \neq [0]$. There is an $n \in \mathbb{N}$ such that $x \in F_n$. Since $\varphi_n(x) = [x] \neq 0$, it follows that for each $m \geq n$, we have $f_n^m(x) \neq 0$, therefore $x \notin \ker\left(f_n^m\right)$. By construction of the direct system, the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F_n & \longrightarrow & C_n & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f_n^{n+1}} & & \downarrow{\scriptstyle g_n^{n+1}} & & \\
0 & \longrightarrow & M & \longrightarrow & F_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f'\varphi_{n+1}} & & \downarrow{\scriptstyle g'\varphi_{n+1}} & & \\
0 & \longrightarrow & M & \longrightarrow & F'' & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

Then,

$$\ker\left(f'\varphi_{n+1}f_n^{n+1}\right) = \ker(f'\varphi_n) = \ker\left(f_n^{n+1}\right),$$

which leads to a contradiction. □

**Lemma 5.3.** With the assumptions of Theorem 5.1, let $0 \to M \xrightarrow{\varphi} F' \xrightarrow{\pi} C' \to 0$ be the Ext-generator constructed in Lemma 5.2. Then $\varphi \colon M \to F'$ is a $\mathscr{C}^\perp$-envelope of $M$.

*Proof.*

- Firstly, we will show that for any commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f'} & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

then $f'$ is an automorphism. Since $f'$ is injective by the previous lemma, it suffices to show that $f'$ is surjective. We assume that this is not true. We set:

$$\begin{aligned}(0 \to M \to F_0 \to C_0 \to 0) &= (0 \to M \to F_1 \to C_1 \to 0) \\ &= (0 \to M \to F \to C \to 0)\end{aligned}$$

and $f_{01} = f'$. Then there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F_0 & \longrightarrow & C_0 & \longrightarrow & 0 \\ & & \Vert & & \downarrow{\scriptstyle f_{01}} & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F_1 & \longrightarrow & C_1 & \longrightarrow & 0 \end{array}$$

Again, we set:

$$(0 \to M \to F_2 \to C_2 \to 0) = (0 \to M \to F \to C \to 0)$$

Then there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \Vert & & \downarrow{\scriptstyle f_{12}} & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F_2 & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

where $f_{12}$ is injective but not surjective. If $f_{02} = f_{12} f_{01}$, then $\mathrm{Im} f_{02} \subsetneqq \mathrm{Im} f_{01}$. In general, for each $n \in \mathbb{N}$, there exist

$$(0 \to M \to F_n \to C_n \to 0) \quad \text{and} \quad f_{n-1,n} \in \mathrm{Hom}\,(F_{n-1}, F_n)$$

where $f_{n-1,n}$ is injective but not surjective, such that

$$\mathrm{Im} f_{0,n} \subsetneqq \mathrm{Im} f_{1,n} \subsetneqq \cdots \subsetneqq \mathrm{Im} f_{n-1,n} \subsetneqq F'.$$

Therefore, we have that $\mathrm{card}\,(F') \geq n$, for all $n \in \mathbb{N}$, so $\mathrm{card}\,(F') \geq \omega$. We shall show that $\mathrm{card}\,(F') \geq \beta$, for all ordinals $\beta$, and that leads to a contradiction.

Let $\beta$ be an arbitrary ordinal. We assume that there exist

$$(0 \to M \to F_\lambda \to C_\lambda \to 0) = (0 \to M \to F \to C \to 0)\,, \quad \forall\, \lambda < \beta$$

and for each $\lambda < \beta$, we have

$$(f_{\lambda,\lambda+1} \colon F_\lambda \to F_{\lambda+1}) = \left(f' \colon F' \to F'\right),$$

and for each $\kappa < \lambda < \nu < \beta$,

$$f_{\kappa,\nu} = f_{\lambda,\nu} f_{\kappa,\lambda}.$$

- If $\beta = \gamma + 1$, then we set

$$(0 \to M \to F_\beta \to C_\beta \to 0) = (0 \to M \to F \to C \to 0)$$

and $f_{\gamma,\beta} = f'$. Then we set $f_{\lambda,\beta} = f_{\gamma,\beta} f_{\lambda,\gamma}$. Then there is a strictly increasing chain of submodules of $F'$:

$$\{\mathrm{Im} f_{\lambda,\beta} \mid \lambda < \beta\}$$

thus, card $(F') \geq \beta$.

- If $\beta$ is a limit ordinal, we take the direct limit of the above direct system:

$$\left(0 \to M \to F'_\beta \to C'_\beta \to 0\right) = \left(0 \to M \to \varinjlim F_\lambda \to \varinjlim C_\lambda \to 0\right)$$

and we set $g_{\lambda,\beta}$ to be the canonical induced maps, for all $\lambda < \beta$. Since the sequence $(0 \to M \to F' \to C' \to 0)$ is an Ext-generator, there exist a pair of morphisms $(h, p)$ and a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F'_\beta & \longrightarrow & C'_\beta & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow h & & \Big\downarrow g & & \\ 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

and we set

$$(0 \to M \to F_\beta \to C_\beta \to 0) = \left(0 \to M \to F' \to C' \to 0\right)$$

and

$$f_{\lambda,\beta} = h g_{\lambda,\beta}.$$

From the above relations, it can be easily shown that

$$\{\mathrm{Im} f_{\lambda,\beta} \mid \lambda < \beta\}$$

is a strictly increasing chain consisting of submodules of $F'$, therefore, card$(F') \geq \beta$.

- Secondly, we will show that $F' \in \mathscr{C}^\perp$. Let $\overline{C} \in \mathscr{C}$. We will show that $\mathrm{Ext}^1_R\left(\overline{C}, F'\right) = 0$. We consider an extension:

$$0 \to F' \xrightarrow{f} E \to \overline{C} \to 0$$

and it suffices to show that the above sequence splits. By the following pushout diagram:

$$\begin{array}{ccc} F' & \longrightarrow & C' \\ \big\downarrow & & \big\downarrow \\ E & \longrightarrow & P \end{array}$$

and Lemma 1.3 (pushout version), we have the following diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \big\| & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & P & \longrightarrow & 0 \\ & & & & \big\downarrow & & \big\downarrow & & \\ & & & & \overline{C} & = & \overline{C} & & \\ & & & & \big\downarrow & & \big\downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since $\mathscr{C}$ is closed under extension, by the fourth column of the above diagram, we have that $P \in \mathscr{C}$. Since $0 \to M \to F' \to C' \to 0$ is an Ext-generator, there exist linear maps $h, \ell$ making the following diagram commutative.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \big\| & & {\scriptstyle h}\big\uparrow\big\downarrow{\scriptstyle f} & & {\scriptstyle \ell}\big\uparrow\big\downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & P & \longrightarrow & 0 \\ & & & & \big\downarrow & & \big\downarrow & & \\ & & & & \overline{C} & = & \overline{C} & & \\ & & & & \big\downarrow & & \big\downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Then we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle hf} & & \downarrow{\scriptstyle \ell g} & & \\ 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

and from the above part, it follows that $hf$ and $\ell g$ are automorphisms. Then, it is clear that the middle column of the above diagram:

$$0 \to F' \xrightarrow{f} E \to \overline{C} \to 0$$

splits, and we are done.

- Finally, we shall show that

$$0 \to M \xrightarrow{\alpha} F' \to C' \to 0$$

is a $\mathscr{C}^{\perp}$-preenvelope, and combined with the first part, we obtain the desired result. Let $F'' \in \mathscr{C}^{\perp}$ and $M \xrightarrow{\gamma} F''$. Since

$$0 \to M \xrightarrow{\beta} F \to C \to 0$$

is a special $\mathscr{C}^{\perp}$-preenvelope, there exists a linear map $\delta$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\beta} & F \\ & {\scriptstyle \gamma}\searrow & \downarrow{\scriptstyle \delta} \\ & & F'' \end{array}$$

Since $0 \to M \xrightarrow{\beta} F \to C \to 0$ is an Ext-generator, there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\alpha} & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\beta} & F & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

The above observations imply the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & F' \\ & \searrow^{\gamma} & \downarrow{\scriptstyle \delta\varphi} \\ & & F'' \end{array}$$

and we are done.

□

We have demonstrated that, in a module class $\mathscr{C}$ closed under extensions and direct limits, the existence of a special preenvelope leads to the existence of a $\mathscr{C}^{\perp}$-envelope. Can the dual result also hold? In other words, does the existence of a $\mathscr{L}$-precover imply the existence of a $\mathscr{L}$-cover?

The answer is yes! The proof of this result can be established by making slight modifications to the preceding lemmas and their proofs. For this reason, we will only state the necessary lemmas required for the proof of the theorem.

**Theorem 5.2** (Xu)**.** Assume $\mathscr{C}$ is closed under direct limits. If $M$ has a $\mathscr{C}$-precover, then $M$ has a $\mathscr{C}$-cover $L \to M$. Furthermore, if $\mathscr{C}$ is closed under extensions, then $\ker(L \to M) \in \mathscr{C}^{\perp}$.

**Lemma 5.4.** Assume $\mathscr{C}$ is closed under direct limits. If $L \to M$ is a $\mathscr{C}$-precover of $M$, then there exists a precover $\overline{L} \to M$ and a commutative diagram:

$$\begin{array}{ccc} L & \longrightarrow & M \\ \downarrow{\scriptstyle f} & & \| \\ \overline{L} & \longrightarrow & M \end{array}$$

such that for any precover $L^* \to M$ and any commutative diagram:

$$\begin{array}{ccc} \overline{L} & \longrightarrow & M \\ \downarrow{\scriptstyle g} & & \| \\ L^* & \longrightarrow & M \end{array}$$

it follows that $\ker(gf) = \ker(f)$.

**Lemma 5.5.** Assume $\mathscr{C}$ is closed under direct limits. If $L \to M$ is a $\mathscr{C}$-precover of $M$, then there exists a precover $\overline{L} \to M$ and a commutative diagram:

$$\begin{array}{ccc} L & \longrightarrow & M \\ \Big\downarrow f & & \Big\| \\ \overline{L} & \longrightarrow & M \end{array}$$

such that for any precover $L^* \to M$ and any commutative diagram:

$$\begin{array}{ccc} \overline{L} & \longrightarrow & M \\ \Big\downarrow g & & \Big\| \\ L^* & \longrightarrow & M \end{array}$$

it follows that $\ker(g) = 0$.

**Lemma 5.6.** Assume $\mathscr{C}$ is closed under direct limits, and let $L \to M$ be a $\mathscr{C}$-precover of $M$. If $\overline{L} \to M$ is the precover defined in the previous lemma, then $\overline{L} \to M$ is an $\mathscr{L}$-cover.

## 5.2 Flat Cotorsion Theory

**Definition 5.2.** Let $M \in R$-Mod. The **character module** of $M$ is defined as

$$\mathrm{D}M = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

It is straightforward to verify that $\mathrm{D}M \in$ Mod-$R$ with the right action given by

$$(f \cdot r)(x) = f(rx), \quad \forall x \in M,\ r \in R.$$

In the case where $M$ is a right $R$-module, $\mathrm{D}M$ is defined analogously and can be endowed with a left $R$-module structure as follows:

$$(r \cdot f)(x) = f(xr), \quad \forall x \in M,\ ,\ \forall r \in R.$$

**Proposition 5.1.** (a) $\mathrm{D}M = 0$ if and only if $M = 0$.

(b) If $\mathbb{X}_* \in \mathrm{Ch}(R)$, then $\mathbb{X}_*$ is acyclic[1] if and only if $\mathrm{D}(\mathbb{X}_*)$ is acyclic.

(c) For each $M \in R$-Mod and $N \in$ Mod-$R$, there is a natural isomorphism of abelian groups:

$$\mathrm{D}(N \otimes_R M) \cong \mathrm{Hom}_R(M, \mathrm{D}N).$$

Similarly, there is a natural isomorphism of abelian groups:

$$\mathrm{D}(N \otimes_R M) \cong \mathrm{Hom}_R(N, \mathrm{D}M).$$

(d) For each $M \in R$-Mod and $N \in$ Mod-$R$, there is a natural isomorphism of abelian groups:

$$\mathrm{D}\left(\mathrm{Tor}_n^R(N, M)\right) \cong \mathrm{Ext}_R^n(M, \mathrm{D}N).$$

Similarly, there is a natural isomorphism of abelian groups:

$$\mathrm{D}\left(\mathrm{Tor}_n^R(N, M)\right) \cong \mathrm{Ext}_R^n(N, \mathrm{D}M).$$

*Proof.* (a) It is obvious that if $M = 0$, then $\mathrm{D}M = 0$. Conversely, we assume that $\mathrm{D}M = 0$, and for the sake of contradiction, we assume that $M \neq 0$. Then there exists $x \in M$ such that $x \neq 0$. Therefore, there is a nontrivial $\mathbb{Z}$-morphism:

$$\varphi \colon \langle x \rangle \to \mathbb{Q}/\mathbb{Z}.$$

Since $\langle x \rangle \hookrightarrow M$ and $\mathbb{Q}/\mathbb{Z}$ is a divisible abelian group, $\varphi$ can be extended to

$$f \colon M \to \mathbb{Q}/\mathbb{Z} \neq 0$$

which is a contradiction.

(b) We assume that $\mathbb{X}_*$ is acyclic. Then,

$$\mathrm{H}_R^n(\mathrm{D}(\mathbb{X}_*)) = \mathrm{D}\left(\mathrm{H}_n^R(\mathbb{X}_*)\right) = \mathrm{D}(0) = 0, \quad \forall n \in \mathbb{N}.$$

Conversely, we assume that $\mathrm{D}(\mathbb{X}_*)$ is acyclic. Similarly, we have that

$$\mathrm{H}_R^n(\mathrm{D}(\mathbb{X}_*)) = \mathrm{D}\left(\mathrm{H}_n^R(\mathbb{X}_*)\right) = 0.$$

By (a), we conclude that $\mathrm{H}_n^R(\mathbb{X}_*) = 0$ for all $n \in \mathbb{N}$.

(c) We consider the mapping:

$$\varphi \colon \mathrm{Hom}_{\mathbb{Z}}(N \otimes_R M, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}))$$

defined by:

$$\varphi(f)(m)(n) = f(n \otimes m) \in \mathbb{Q}/\mathbb{Z}.$$

[1]A chain complex $\mathbb{X}_*$ is acyclic if $\mathrm{H}_n^R(\mathbb{X}_*) = 0$ for all $n \in \mathbb{N}$.

(d) Let $\mathbb{P}_* \to M$ be a projective resolution of $M$. Then:

$$\begin{aligned} \mathrm{D}\left(\mathrm{Tor}_n^R(N,M)\right) &= \mathrm{D}\left(\mathrm{H}_n^R\left(N \otimes_R \mathbb{P}_*\right)\right) \\ &= \mathrm{H}_R^n\left(\mathrm{D}(N \otimes_R \mathbb{P}_*)\right) \\ &\cong \mathrm{H}_R^n\left(\mathrm{Hom}\left(\mathbb{P}_*, \mathrm{D}N\right)\right), \end{aligned}$$

where:

$$\mathrm{Ext}_R^n\left(M, \mathrm{D}N\right) = \mathrm{H}_R^n\left(\mathrm{Hom}\left(\mathbb{P}_*, \mathrm{D}N\right)\right).$$

□

**Definition 5.3.** For a class (right resp. left) of $R$ modules $\mathscr{C}$ we put

$$\mathscr{C}^\top = \left\{N \in R\text{-Mod} \mid \mathrm{Tor}_1^R(C,N) = 0 \text{ for all } C \in \mathscr{C}\right\}$$

$$^\top\mathscr{C} = \left\{N \in R\text{-Mod} \mid \mathrm{Tor}_1^R(N,C) = 0 \text{ for all } C \in \mathscr{C}\right\}$$

**Definition 5.4.** Let $\mathscr{A}, \mathscr{B} \subseteq \text{Mod-}R$. The pair $(\mathscr{A}, \mathscr{B})$ is called a **Tor - torsion theory** if

$$\mathscr{A} =^\top \mathscr{B} \quad \text{and} \quad \mathscr{B} = \mathscr{A}^\top.$$

**Example 5.1.** Using Chapter 3 it can be easily seen that the pair $(\mathscr{FL}(R), R\text{-Mod})$, where $\mathscr{FL}(R)$ is the class of flat $R$ - modules, is a Tor - torsion theory.

**Lemma 5.7.** Let $R$ be a ring and $(\mathscr{A}, \mathscr{B})$ be a Tor - torsion theory. Then $= \left(\mathscr{A}, \mathscr{A}^\perp\right)$ is a cotorsion theory.

*Proof.* We want to show that $\mathscr{A} =^\perp \left(\mathscr{A}^\perp\right)$. The relation $\mathscr{A} \subseteq^\perp \left(\mathscr{A}^\perp\right)$ is obvious. Conversely, let $M \in^\perp \left(\mathscr{A}^\perp\right)$ and we want to show that $M \in \mathscr{A} =^\top \mathscr{B}$. Let $B \in \mathscr{B}$ and by Proposition 5.1 there is a canonical isomorphism

$$\mathrm{D}\left(\mathrm{Tor}_1^R(M,B)\right) \cong \mathrm{Ext}_R^1\left(M, \mathrm{D}B\right)$$

By Proposition 5.1 (a), it suffices to show that $\mathrm{D}B \in \mathscr{A}^\perp$. Similarly, if $A \in \mathscr{A}$ then

$$\mathrm{Ext}_R^1\left(A, \mathrm{D}B\right) \cong \mathrm{D}\left(\underbrace{\mathrm{Tor}_1^R(A,B)}_{0}\right) = 0$$

and we are done. □

**Definition 5.5.** By previous lemma it can be easily seen that $(\mathscr{FL}(R), R\text{-Mod})$ is a Tor - torsion pair, therefore the pair $(\mathscr{FL}(R), \mathscr{E}(R))$ is a cotorsion pair, where $\mathscr{E}(R) = \mathscr{FL}(R)^\perp$ is called the class of all **Enochs cotorsion** modules .

## 5.3 The Class of Flat Modules is a Cover Class

**Lemma 5.8.** Let $R$ be a ring and $M \in R$-Mod. Let $\kappa$ be an infinite cardinal such that $|R| \leq \kappa$ (for example, $\kappa = \max\{|R|, \aleph_0\}$). If $X \subseteq M$ such that $|X| \leq \kappa$, then there is a **pure** submodule $N \subseteq M$ such that

$$X \subseteq N \quad \text{and} \quad |N| \leq \kappa.$$

*Proof.* If we set $N = \langle X \rangle = \sum_{x \in X} Rx$, then it is easy to see that

$$N = \sum_{x \in X} Rx = \bigcup_{X' \subseteq X, |X'| < \infty} \left( \sum_{x \in X'} Rx \right).$$

Since $|R| \leq \kappa$, then

$$\left| \sum_{x \in X'} Rx \right| \leq \left| R^{|X'|} \right| = |R| \leq \kappa, \quad \text{for all } X' \subseteq X \text{ such that } |X'| < \infty,$$

and since $|X| \leq \kappa$, we have that $|N| \leq \kappa$. By the definition of $N$, it is not clear whether $N$ is pure. According to Theorem 3.2, $N$ is pure if and only if for every commutative diagram

$$\begin{array}{ccc} R^n & \xrightarrow{\alpha} & R^m \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} \\ N & \xrightarrow{i} & M \end{array}$$

there is a linear map $\varphi\colon R^m \to N$ such that the following diagram commutes:

$$\begin{array}{ccc} R^n & \xrightarrow{\alpha} & R^m \\ \Big\downarrow{\scriptstyle f} & \swarrow{\scriptstyle \varphi} & \Big\downarrow{\scriptstyle g} \\ N & \xrightarrow{i} & M \end{array}$$

Based on this requirement, and since we cannot ensure that $N$ is pure, we proceed with the following construction for any submodule $N$ of $M$ where $|N| \leq \kappa$. First, let

$$\mathscr{X} = \{f\colon R^n \to N \mid f \text{ is an } R\text{-homomorphism}, \ n \in \mathbb{N}\}.$$

For each $f \in \mathscr{X}$, we define:

$$\mathscr{I}_f = \{\alpha\colon R^n \to R^m \mid \exists\, g\colon R^m \to M \ : \ g\alpha = if, \ m \in \mathbb{N}\}.$$

Since each $f \in \mathscr{X}$ is determined by the image of $n$ generators, it is easy to see that

$$\mathscr{X} \equiv \bigsqcup_{n=1}^{\infty} N^n \Rightarrow |\mathscr{X}| \leq \aleph_0 \cdot \kappa = \kappa.$$

In addition, it is a straightforward verification that $|\mathscr{J}_f| \leq \kappa$, for all $f \in \mathscr{X}$. We set

$$Z_N = \{(f, \alpha) \mid f \in \mathscr{X},\ \alpha \in \mathscr{J}_f\}.$$

From the above remarks, we have $|Z_N| \leq \kappa$. For each $(f, \alpha) \in Z_N$, we choose a $g\colon R^m \to M$ such that $g\alpha = if$. We then consider the following diagram:

$$\begin{array}{ccc} \bigoplus_{(f,\alpha)\in Z_N} R^n & \xrightarrow{A} & \bigoplus_{(f,\alpha)\in Z_N} R^m \\ \downarrow F & & \downarrow G \\ N & \xrightarrow{i} & M \end{array}$$

where $F \circ \nu_{(f,\alpha)} = f$ and $G \circ p_{(f,\alpha)} = g_{(f,\alpha)}$, if

$$\nu_{(f,\alpha)}\colon R^n \hookrightarrow \bigoplus_{(f,\alpha)\in Z_N} R^n \quad \text{and} \quad p_{(f,\alpha)}\colon R^m \to \bigoplus_{(f,\alpha)\in Z_N} R^m$$

are the canonical embeddings.

We define

$$N' = N + \mathrm{Im}(G) = N + \sum_{(f,\alpha)\in Z_N} g_{(f,\alpha)}(R^m)$$

We can make the following observations.

- Since $\left|g_{(f,\alpha)}(R^m)\right| \leq |R^m| \leq \kappa$, $|Z_N| \leq \kappa$, and

$$N' = N + \sum_{(f,\alpha)\in Z_N} g_{(f,\alpha)}(R^m) = N + \bigcup_{Z' \subseteq Z,\ |Z'|<\infty} \left( \sum_{(f,\alpha)\in Z'} g_{(f,\alpha)}(R^m) \right)$$

  then we have that $|N'| \leq \kappa$.

- For every commutative diagram

$$\begin{array}{ccc} R^n & \xrightarrow{\alpha} & R^m \\ \downarrow f & & \downarrow g \\ N & \xrightarrow{i} & M \end{array}$$

there is a commutative diagram

$$\begin{CD} R^n @>{\alpha}>> R^m \\ @V{f}VV @VV{g}V \\ N @>{i}>> N' \end{CD}$$

Based on the previous construction, we inductively define a sequence of submodules of $M$ as follows: let $N_0 = \langle X \rangle$, and for each $n \in \mathbb{N}$, define $N_{n+1} := (N_n)'$ as described above. From the construction, it follows that $|N_n| \leq \kappa$ for all $n \in \mathbb{N}$. We then define

$$N = \bigcup_{n=0}^{\infty} N_n$$

and proceed to show that $N$ satisfies the desired properties.

- By the definition of $N$, we have that $X \subseteq N$ and $|N| \leq \kappa$.
- We will now show that $N$ is pure. Consider the following commutative diagram:

$$\begin{CD} R^n @>{\alpha}>> R^m \\ @V{f}VV @VV{g}V \\ N @>{i}>> M \end{CD}$$

  Since $R^n$ is finitely generated, there exists a $k \in \mathbb{N}$ such that $f(R^n) \subseteq N_k$. If we identify $f$ with $f_k \colon R^n \to N_k$, then it holds that $f = \tau_k f_k$, where $\tau_k \colon N_k \hookrightarrow N$ is the inclusion map. Consequently, there exists a $g_k \colon R^m \to N_{k+1}$ such that the following diagram commutes:

$$\begin{CD} R^n @>{\alpha}>> R^m \\ @V{f_k}VV @VV{g_k}V \\ N_k @>{i}>> N_{k+1} \end{CD}$$

  If we set $\varphi := \tau_{k+1} \circ g_k \colon R^m \to N$, then the following diagram commutes:

$$\begin{array}{ccc} R^n & \xrightarrow{\alpha} & R^m \\ \Big\downarrow f & \overset{\varphi}{\swarrow} & \Big\downarrow g \\ N & \xrightarrow{i} & M \end{array}$$

Thus, the proof is complete.

□

**Proposition 5.2.** Let $M$ be an $R$-module and $\kappa$ be an infinite cardinal such that $|R| < \kappa$. Then there exists an ordinal $\lambda$ and a continuous increasing family of pure submodules $\{M_\alpha \mid \alpha < \lambda\}$ such that:

- $M_0 = 0$,
- $M = M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$,
- $|M_{\alpha+1}/M_\alpha| \leq \kappa$, for all $\alpha + 1 < \lambda$.

*Proof.* We enumerate $M$ as $M = \{x_\alpha \mid \alpha < \lambda\}$, where $|M| = \lambda$. We will define a family $\{M_\alpha \mid \alpha < \lambda\}$ that satisfies the desired properties using induction.

- **Base Case.** $M_0 = 0$.
- **Successor Case.** Let $\alpha = \gamma + 1$. We assume that $M_\beta$ has been defined for every $\beta < \alpha$ such that:

  $$|M_{\beta+1}/M_\beta| \leq \kappa, \quad \text{for every } \beta + 1 < \alpha,$$

  and $M_\beta = \bigcup_{\beta'<\beta} M_{\beta'}$, whenever $\beta$ is a limit ordinal.

  If $M_\gamma = M$, then the procedure stops, and we are done. Otherwise, let $x \in M \setminus M_\gamma$, therefore

  $$0 \neq R\overline{x} \subseteq M/M_\gamma.$$

  Since $|R\overline{x}| \leq |R| \leq \kappa$, then by Lemma 5.8, there exists a pure submodule $N \subseteq M/M_\gamma$ such that $R \cdot \overline{x} \subseteq N$ and $|N| \leq \kappa$.

  Then, there exists a submodule $M_{\gamma+1} \subseteq M$ such that $M_\gamma \subseteq M_{\gamma+1}$ and

  $$N = M_{\gamma+1}/M_\gamma \Rightarrow |M_{\gamma+1}/M_\gamma| = |N| \leq \kappa.$$

  Finally, we will show that $M_{\gamma+1}$ is pure, which follows immediately since

  $$N = M_{\gamma+1}/M_\gamma \subseteq M/M_\gamma \quad \text{and} \quad M_\gamma \subseteq M$$

  are pure.
- **Limit Case.** If $\alpha$ is a limit ordinal, then we define $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$, and $M_\alpha$ is pure.

□

**Corollary 5.1.** If $M \in \mathscr{FL}(R)$ and $\kappa$ is an infinite cardinal such that $|R| < \kappa$, then there exists a continuous, increasing family of flat submodules satisfying:

- $M_0 = 0$
- $M_{\alpha+1}/M_\alpha$ is flat and $|M_{\alpha+1}/M_\alpha| \leq \kappa$
- $M = \bigcup_\alpha M_\alpha$

*Proof.* We consider a pure family $\{M_\alpha\}_\alpha$ as in the previous proposition. Since $M_\alpha \subseteq M$ is pure and $M$ is flat, by Corollary 3.3, it follows that $M_\alpha$ is flat.

Since $M_\alpha \subseteq M_{\alpha+1}$ is pure, we consider the following pure exact sequence:

$$0 \to M_\alpha \to M_{\alpha+1} \to M_{\alpha+1}/M_\alpha \to 0$$

By Corollary 3.3, the quotient $M_{\alpha+1}/M_\alpha$ is flat. □

**Theorem 5.3.** The flat cotorsion theory $(\mathscr{FL}(R), \mathscr{E}(R))$ is complete. Since $\mathscr{FL}(R)$ is closed under direct limits, it follows from Theorem 5.2 that $\mathscr{FL}(R)$ is a cover class.

*Proof.* We consider the set

$$\mathscr{S} = \{M \text{ is flat} \mid |M| \leq \kappa\}.$$

We will show that the flat cotorsion theory $(\mathscr{FL}(R), \mathscr{E}(R))$ is cogenerated by $\mathscr{S}$, which is equivalent to

$$\mathscr{E}(R) = (\mathscr{FL}(R))^\perp = \mathscr{S}^\perp.$$

The first relation is obvious since

$$\mathscr{S} \subseteq \mathscr{FL}(R) \Rightarrow \mathscr{E}(R) = (\mathscr{FL}(R))^\perp \subseteq \mathscr{S}^\perp.$$

Let $C \in \mathscr{S}^\perp$. We shall show that

$$\operatorname{Ext}^1(F, C) = 0, \quad \text{for all } F \in \mathscr{FL}(R).$$

Let $F \in \mathscr{F}\mathscr{L}(R)$ and $\{F_\alpha \mid \alpha < \lambda\}$ be a flat family of submodules with the properties of the above corollary. Then,

$$F_{\alpha+1}/F_\alpha \in \mathscr{F}\mathscr{L}(R) \quad \text{and} \quad |F_{\alpha+1}/F_\alpha| \leq \kappa,$$

therefore, $F_{\alpha+1}/F_\alpha \in \mathscr{S}$. Since

$$\operatorname{Ext}^1(F_0, C) = 0 \quad \text{and} \quad \operatorname{Ext}^1(F_{\alpha+1}/F_\alpha, C) = 0, \quad \text{for all } \alpha + 1 < \lambda,$$

by Lemma 4.5, we deduce that

$$\operatorname{Ext}^1(F, C) = \operatorname{Ext}^1\left(\bigcup_{\alpha<\lambda} F_\alpha, C\right) = 0.$$

□

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